Rational Elliptic Surfaces over the Rational Numbers with Geometric Mordell-Weil Rank 7

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Abstract

Over the past years, several authors have given all equivalent constructions of rational elliptic surfaces with Mordell-Weil ranks between 8 and 4 over an algebraically closed field of characteristic zero, via the description of pencils of plane cubics (see [Shi91], [Fus06], [Sal09], [Pas10]). This classification via pencils of cubics is possible thanks to a theorem of Miranda which states that, over an algebraically closed field, every relatively minimal rational elliptic surface is isomorphic to the blow-up of the plane on the base points of a pencil of cubics (see [Mir89]). This fact does not hold true over an arbitrary field. Indeed, its validity would imply that all rational elliptic surfaces defined over an arbitrary field are rational over that field, which we know is not true as, for instance, they can be obtained by the blow up of the unique base point of the anti-canonical linear system of a k-minimal del Pezzo surface of degree one which is known to be irrational over k. It is expected then that surfaces that were equivalent over an algebraically closed field may be no longer equivalent over a non-algebraically closed field over which they are defined. The goal of this dissertation is to explore this feature and provide different non-equivalent constructions of rational elliptic surfaces of Mordell-Weil rank 7 over the rational numbers, providing examples that are $\mathbb{Q}$-rational and non-equivalent, and examples of $\mathbb{Q}$-irrational rational elliptic surfaces.

Resumo

Nos últimos anos, diversos autores mostraram todas as construções equivalentes de superfícies elípticas racionais com posto de Mordell-Weil entre 8 e 4 sobre um corpo algébricamente fechado de característica zero, através da descrição de *pencils* de cúbicas planas (veja [Shi91], [Fus06], [Sal09], [Pas10]). Esta classificação através de *pencils* de cúbicas é possível graças a um resultado de Mirandaque afirma que, sobre um corpo algebricamente fechado, toda superfície elíptica racional relativamente minimal é isomorfa ao *blow-up* do plano nos pontos de base de um pencil de cúbicas (veja [Mir89]). Esse fato não se mantém verdadeiro sobre um corpo arbitrário. De fato, sua validade implicaria que toda superfície elíptica racional definidas sobre um corpo arbitrário são racionais sobre este corpo, o que nós sabemos não ser verdade pois, por exemplo, elas podem ser obtidas pelo *blow-up* de um único ponto de base do sistema linear anti-canônico de uma superfície de del Pezzo k-minimal de grau 1, que sabemos ser irracional sobre k. É esperado que superfícies que eram equivalentes sobre um corpo algebricamente fechado possam não ser mais equivalentes sobre o corpo não algebricamente fechado sobre o qual elas estão definidas. O objetivo desta dissertção é explorar este acontecimento e mostrar diferentes construções não-equivalentes de superfícies elípticas racionais sobre os números racionais com posto de Mordell-Weil 7, dando exemplos que são $\mathbb{Q}$-racionais mas não são equivalentes, e exemples de superfícies elípticas racionais que são $\mathbb{Q}$-irracionais.

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Notation

A_r, D_r, E_r : Root Lattices.
$\mathbb{P}^n$: Projective Space.
$\mathrm{Div}(X)$: Divisor group of a variety X.
$\mathrm{Pic}(X)$: Picard group of a variety X.
$\mathrm{NS}(X)$: Néron-Severi group of a variety X.
$\mathrm{Div}(X)_k$: Subgroup of elements of $\mathrm{Div}(X)$ invariant under $\mathrm{Gal}(\overline{k}/k)$.
$\mathrm{Pic}(X)_k$: Subgroup of elements of $\mathrm{Pic}(X)$ invariant under $\mathrm{Gal}(\overline{k}/k)$.
$\mathrm{NS}(X)_k$: Subgroup of elements of $\mathrm{NS}(X)$ invariant under $\mathrm{Gal}(\overline{k}/k)$.
$\rho(S)$: Rank of $\mathrm{NS}(S)$.
$\rho(S)_k$: Rank of $\mathrm{NS}(S)_k$.
r : Rank of $E(\overline{k}(C))$.
r_k : Rank of $E(k(C))$.

Contents

Introduction

Let k be a number field. An elliptic surface S defined over k (see Def. 2.1.1) is a smooth projective algebraic surface over k together with a k-morphism $\pi : S \to C$ to a base curve C over k, such that almost all fibers $\pi^{-1}(v)$ are elliptic curves inside S. Only finitely many fibers are not elliptic curves. Such fibers are singular and may be reducible. In this text, we assume moreover, that elliptic surfaces are not of product type, i.e., they are not isomorphic to $E \times C$ with E an elliptic curve over k.

The generic fiber of π is an elliptic curve E over the function field $K = k(C)$. The K-points of E are in a one-to-one relation with the sections $\sigma : C \to S$ of the elliptic surface, and the group $E(K)$ is finitely generated.

Let $\overline{k}$ be an algebraic closure of k, which will be fixed once and for all. Then, over $\bar{k}$, the Shioda-Tate formula (2.2.5) relates the rank of E, denoted by r, the rank of the Néron-Severi group of S, denoted by ρ, and the quantity of different components in each fiber, denoted by m_v:

$$\rho = 2 + r + \sum_{v \in C} (m_v - 1).$$

In the setting of the previous paragraph, a result of Miranda ([Mir89, Lem. IV.1.2]) tells us that when S is rational then it is isomorphic to the blow up of $\mathbb{P}^2$ in the 9 base points of a pencil of cubics. Note that, in this case, the base curve is the projective line and the Néron-Severi group has rank 10, giving us a direct relation between the Mordell-Weil rank and the reducible fibers:

$$r = 8 - \sum_{v \in C} (m_v - 1).$$

If we fix $r = 7$, then, by the formula above, π admits exactly one reducible fiber, which in turn has two components. The latter can be of type I_2, when the components meet transversally in two points, or of type III, when they meet tangentially in a unique point (see the Kodaira classification of reducible fibers in Thm. 2.1.11).

Working over $\mathbb{C}$, Fusi shows in [Fus06] that given the type of reducible fiber with two components, there is a unique construction of the surface via pencil of cubics modulo equivalence. We say that two constructions are equivalent if there is a series of Cremona transformations in the plane that sends one pencil of cubics to the other. Over the number field k, we say that two constructions are k-equivalent if the Cremona maps can be taken over k. Fusi and others ([Fus06], [Sal09], [Pas10]) studied $\mathbb{C}$-equivalent constructions of rational elliptic surfaces of ranks $4 \leq r \leq 8$.

The aim of this dissertation is to study $\mathbb{Q}$-equivalent constructions with Mordell-Weil rank 7 over $\overline{\mathbb{Q}}$. The text is organized as follows.

The first two chapters are dedicated to covering the background needed to tackle the constructions of rational elliptic surfaces. The reader with a background on basic algebraic geometry may skip the first chapter. Those with knowledge of the basic theory of elliptic surfaces may skip the second chapter.

The first chapter introduces the basic theory and tools required for this dissertation. Section 1.1 introduces the theory of lattices and defines the root lattices A_r, D_r, E_r that will appear as Mordell-Weil lattices of rational elliptic surfaces. Section 1.2 introduces the basic theory of Algebraic Geometry, based on [Har77] and [Sha77], but dealing with fields that are not algebraically closed. We also define the Néron-Severi group and the Picard number of a variety, which will be important for the definition of the Mordell-Weil lattice of an elliptic surface. In Section 1.3 we show some basic results of algebraic curves that are implicitly used, such as Bézout's theorem for plane curves. Later, we give the basic definitions of elliptic curves. Section 1.4 contains a brief introduction to algebraic surfaces, as well as tools for showing k-equivalence of different construction of rational elliptic surfaces: namely the Cremona transformations and the theory of k-minimal surfaces.

The second chapter introduces the main subject of the dissertation, namely elliptic surfaces. In Section 2.1 the definition and basic properties of elliptic surfaces are given, along with Kodaira's classification of possible singular fibers. Section 2.2 shows the relation between the Néron-Severi group, the reducible fibers and the generic fiber of an elliptic surface (see 2.2.5). Later, it shows the construction of the Mordell-Weil lattice, by embedding the generic fiber $E(K)$ inside $NS(S) \otimes \mathbb{Q}$. In Section 2.3, we apply the results of the previous sections to rational elliptic surfaces, which allows us to relate the rank of the Mordell-Weil lattice directly to the reducible fibers of the surface.

On the third chapter, we look specifically at rational elliptic surfaces defined over $\mathbb{Q}$. This chapter is dedicated to the study of $\mathbb{Q}$-equivalent and $\mathbb{Q}$-inequivalent constructions of rational elliptic surfaces with Mordell-Weil rank 7 over $\overline{\mathbb{Q}}$. Some constructions that are equivalent over $\overline{\mathbb{Q}}$ are not equivalent over $\mathbb{Q}$. The $\mathbb{Q}$-equivalences depend not only on the geometry of the surfaces, but also on their arithmetic. For example, if two rational elliptic surfaces have $\mathbb{Q}$-equivalent constructions, the rank of their generic fibers must be equal.

We give a classification of the constructions of rational elliptic surfaces with Mordell-Weil rank 7 coming from pencils of cubics defined over $\mathbb{Q}$. In contrast with the geometric case, not every rational elliptic surface with Mordell-Weil rank 7 can be constructed by cubic pencils: surfaces that are rational but not $\mathbb{Q}$-rational provide a clear counter-example. Using a criterion (given for an arbitrary field k) for when a surface is k-rational based on its k-minimal model by [Isk80], we have a way of determining a sufficient condition for a rational elliptic surface to be $\mathbb{Q}$-rational (see Thm. 2.3.8).

We also give several different explicit examples of constructions of $\mathbb{Q}$-rational elliptic surfaces coming from a pencil of cubics, and an example of a rational elliptic surface that is not $\mathbb{Q}$-rational, based on the work of Kuwata in [Kuw05].

Chapter 1

Preliminaries

1.1 Lattices

This section will introduce basic definitions of the theory of lattices, and describe the root lattices A_r, D_r, E_r. It is based on Chapter 2 of [SS17].

1.1.1 Basic Definitions

Definition 1.1.1. A *lattice* L is a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear pairing $\langle \cdot,\cdot \rangle : L \times L \to \mathbb{R}$ such that, extending $\langle \cdot,\cdot \rangle$ to $(L \otimes \mathbb{R}) \times (L \otimes \mathbb{R})$ naturally, if $\langle x, y \rangle = 0$ for all $y \in L \otimes \mathbb{R}$, then $x = 0$. In other words, $\langle \cdot,\cdot \rangle$ is non-degenerate.

Example 1.1.2. (Square and hexagonal lattices)

i) The simplest example of a lattice is the module $\mathbb{Z}^2$ with the natural pairing $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$, called the *square lattice*. The same pairing endows $\mathbb{Z}^r$ with a lattice structure for any $r \in \mathbb{N}$.

ii) Let $\omega = \frac{-1+\sqrt{-3}}{2}$. The module $H = \{a + b\omega | a, b \in \mathbb{Z}\} \subset \mathbb{C}$ can be endowed with a lattice structure with the same pairing as (i), by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

Two lattices L, L' are *isomorphic* if there exists an isomorphism of $\mathbb{Z}$-modules $\varphi : L \to L'$ such that

$$\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle \quad \forall\, x, y \in L.$$

Definition 1.1.3. The *opposite lattice* of a lattice L, denoted by L^-, is the same module L endowed with the opposite pairing $-\langle \cdot,\cdot \rangle$.

Let L be a rank r lattice with a basis $\{e_1, ..., e_r\}$. Then, for $x, y \in L$, we can write:

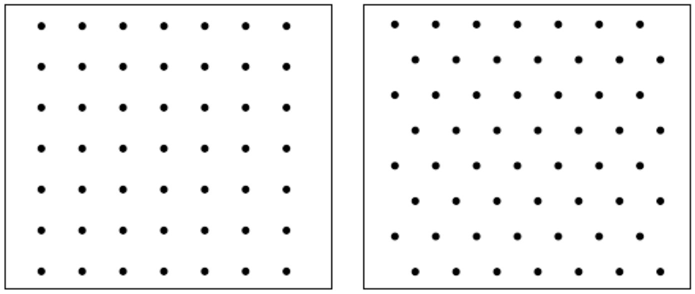


Figure 1: Square lattice and hexagonal lattice.

$$x = \sum_i x_i e_i,\ \ y = \sum_i y_i e_i \quad x_i, y_i \in \mathbb{Z}.$$

Now we can see the pairing in terms of $\langle e_i, e_j \rangle$:

$$\langle x, y \rangle = \sum_{i,j=1}^{r} \langle e_i, e_j \rangle x_i y_j.$$

The matrix $I = (\langle e_i, e_j \rangle)_{i,j}$ is called the *Gram matrix* of L. If we construct the Gram matrix I' for another base of L, then $I' = U^t I U$ for some $U \in GL_r(\mathbb{Z})$.

Definition 1.1.4. The *determinant* of L is defined by $\det L = \det I$.

This definition does not depend on the choice of basis, as $\det I' = (\det U)^2 \det I = \det I$ for any $U \in GL_r(\mathbb{Z})$.

Example 1.1.5. (Square and hexagonal lattices cont.)

i) Taking the natural basis $(1, 0)$ and $(0, 1)$ for the square lattice, we get the Gram matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(\mathbb{Z}^2) = 1.$$

ii) With the basis $\{1, \omega\} \subset H$, we get:

$$I = \begin{pmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{pmatrix} \text{ and } \det(H) = 3/4.$$

Definition 1.1.6. A lattice L is called *integral* if $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in L$. An integral lattice L is called *unimodular* if $\det L = \pm 1$. In our examples, $\mathbb{Z}^2$ is both unimodular and integral, while H is not integral.

Definition 1.1.7. Given L a lattice, a *sublattice* of L is a submodule $T \subset L$ such that the restriction of $\langle \cdot, \cdot \rangle$ to T is non-degenerate. If T is of finite index in L, then $\det T = \det L[L : T]^2$. If L/T is torsion-free, then T is called a *primitive* sublattice.

Definition 1.1.8. We define the *orthogonal complement* of T as

$$T^{\perp} := \{x \in L \mid \langle x, y\rangle = 0,\ \forall y \in T\}.$$

Definition 1.1.9. Given a lattice L, the *dual lattice* of L is defined as

$$L^{\vee} := \{x \in L \otimes \mathbb{Q} \mid \langle x, y\rangle \in \mathbb{Z},\ \forall y \in L\}.$$

We have that L is a sublattice of $L^{\vee}$, and:

$$[L^{\vee} : L] = |\det L|, \qquad \det L^{\vee} = \tfrac{1}{\det L}.$$

Example 1.1.10. Let $L = 2\mathbb{Z}^2$ with the usual pairing. Then, $L^{\vee} = \frac{1}{2}\mathbb{Z}^2$, and we have $\det L = [L^{\vee} : L] = 16$, $\det L^{\vee} = \frac{1}{16}$.

1.1.2 Root Lattices

Definition 1.1.11. A lattice L is called *even* if $\langle x, x\rangle \in 2\mathbb{Z}$, it is called *positive definite*, resp. *negative definite*, if $\langle x, x\rangle > 0$ for all $x \in L$, resp. if $\langle x, x\rangle < 0$.

Definition 1.1.12. Given a definite even lattice L, an element $x \in L$ such that $\langle x, x\rangle = \pm 2$ is called a *root* of L, and $\mathcal{R}(L)$ denotes the set of roots of L. If L is generated by $\mathcal{R}(L)$ then it is called a *root lattice*.

Theorem 1.1.13. *Let L be a positive definite root lattice of rank r. Then, there exists a basis of L, $\{\alpha_1, ..., \alpha_r\} \subset \mathcal{R}(L)$, such that for $i \neq j$:*

$$\langle \alpha_i, \alpha_j\rangle = -1 \text{ or } 0.$$

Definition 1.1.14. Given a positive definite root lattice L with a basis as in the theorem above, we say that L is of type A_r, D_r or E_r if:

$$\begin{aligned}
&(A_r) && \langle \alpha_i, \alpha_j\rangle = -1 \Leftrightarrow && i + 1 = j \\
&(D_r) && \langle \alpha_i, \alpha_j\rangle = -1 \Leftrightarrow && i + 1 = j < r \text{ or } i = r - 2,\ j = r \\
&(E_r) && \langle \alpha_i, \alpha_j\rangle = -1 \Leftrightarrow && i + 1 = j < r \text{ or } i = 3,\ j = r
\end{aligned}$$

We can visualize root lattices by drawing a graph with $\{\alpha_1, ..., \alpha_r\}$ as the vertices and joining α_i and α_j with an edge when $\langle \alpha_i, \alpha_j\rangle = -1$. This is called the *Dynkin diagram* of the lattice.

The determinants for each root lattice are:

	A_r	D_r	E_6	E_7	E_8
det	$r+1$	4	3	2	1

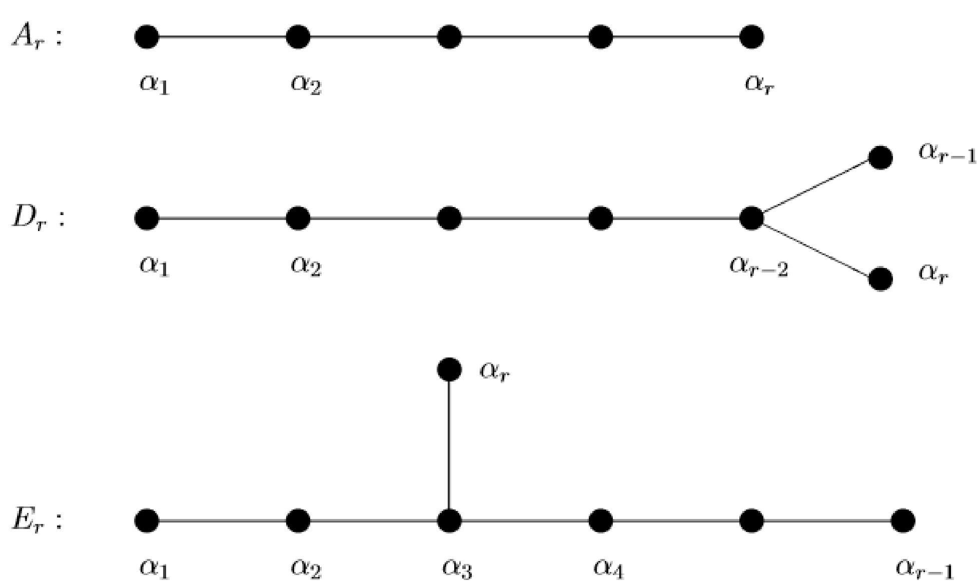


Figure 2: Dynkin diagrams of the root lattices. This figure was taken from [SS17], page 33.

1.2 Algebraic Varieties

1.2.1 Affine and Projective Varieties

Let k be a number field and $\overline{k}$ its algebraic closure, and denote by $G = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of the extension.

Definition 1.2.1. The *affine n-space* over $\overline{k}$ is defined as the set of $n-$tuples:

$$\mathbb{A}^n = \mathbb{A}^n_{\overline{k}} := \{(x_1, \cdots, x_n)|\ x_i \in \overline{k}\}$$

and the set of *k-rational points* of $\mathbb{A}^n$ is defined by:

$$\mathbb{A}^n(k) = \mathbb{A}^n_k := \{(x_1, \cdots, x_n)|\ x_i \in k\}.$$

The Galois group G acts on $\mathbb{A}^n$ by taking a point $P = (x_1, \cdots, x_n)$ to $\sigma(P) := (\sigma(x_1), \cdots, \sigma(x_n))$ for each $\sigma \in G$. This allows us to define $\mathbb{A}^n_k$ equivalently as the set of points of $\mathbb{A}^n$ that are invariant under the action of G.

Definition 1.2.2. The *projective n-space* over $\overline{k}$ is the set of all lines going through the origin in $\mathbb{A}^{n+1}$. Formally, we define an equivalence relation between points in $\mathbb{A}^{n+1} \setminus \{(0, \cdots, 0)\}$ given by $(x_0, \cdots, x_n) \sim (y_0, \cdots, y_n)$ if and only if $(y_0, \cdots, y_n) = \lambda(x_0, \cdots, x_n)$ for some $\lambda \in \overline{k}$; and we can describe the projective space as the quotient:

$$\mathbb{P}^n = \mathbb{P}^n_{\overline{k}} := \frac{\mathbb{A}^{n+1} \setminus \{(0, \cdots, 0)\}}{\sim}.$$

The equivalence class of $(x_0, ..., x_n)$ is called a point in $\mathbb{P}^n$ and is denoted by $[x_0 : \cdots : x_n]$. The set of k-rational points of $\mathbb{P}^n$ is defined by

$$\mathbb{P}^n(k) = \mathbb{P}^n_k := \{P \in \mathbb{P}^n \mid \sigma(P) = P, \ \forall\, \sigma \in G\}.$$

Given an ideal $I \in \overline{k}[x_1, \cdots, x_n]$, we can associate to it a set $V(I) \subset \mathbb{A}^n$ defined by $V(I) := \{p \in \mathbb{A}^n | \ f(p) = 0 \text{ for all } f \in I\}$. Similarly, if I_0 is a homogeneous ideal of $\overline{k}[x_0, \cdots, x_n]$, that is, an ideal generated by homogeneous polynomials, we assign to I_0 a subset of $\mathbb{P}^n$ defined by $V(I_0) := \{p \in \mathbb{P}^n_{\overline{k}} | \ f(p) = 0 \text{ for all } f \in I_0\}$.

On the other way, given a subset $X \subset \mathbb{A}^n$, we define its generating ideal $I(X) := \{f \in \overline{k}[x_1, \cdots, x_n] | f(p) = 0, \ \forall\, p \in X\}$. Similarly, for every subset $Y \subset \mathbb{P}^n_{\overline{k}}$ we define a homogeneous ideal $I(Y) := \langle\{f \in \overline{k}[x_0, ..., x_n] | \ f \text{ is homogeneous}, \ f(p) = 0, \ \forall\, p \in Y\}\rangle$.

Definition 1.2.3. A subset $X \subset \mathbb{A}^n$ is called an *affine algebraic set* if $X = V(I)$ for some ideal $I \subset \overline{k}[x_1, \cdots, x_n]$, and $Y \subset \mathbb{P}^n$ is called a *projective algebraic set* if $Y = V(I_0)$ for some homogeneous ideal $I_0 \in \overline{k}[x_0, \cdots, x_n]$. We endow $\mathbb{A}^n$ and $\mathbb{P}^n$ with a topology, called the *Zariski Topology*, by taking algebraic sets as the closed sets.

We say that an algebraic set X is *defined over* k if it is invariant under G, that is, if for all $P \in X$ and $\sigma \in G$, we have that $\sigma(P) \in X$.

Definition 1.2.4. An algebraic set X is called *reducible* if there exists Y_1, Y_2 algebraic sets such that $Y_1, Y_2 \subsetneq X$ and $X = Y_1 \cup Y_2$. Otherwise, X is *irreducible*, and is called an *algebraic variety* (X is an affine variety if it is an affine algebraic set, and a projective variety if it is a projective algebraic set).

Hilbert's *Nullstellensatz* ensures that the maps $I \mapsto V(I)$ and $V \mapsto I(V)$ define a one-to-one correspondence between algebraic sets and radical ideals, and furthermore algebraic varieties are in a one-to-one correspondence with prime ideals (see [Mat86, Thm. 5.4]).

Notice that a variety V can be viewed as a topological space by taking the subspace topology from $\mathbb{A}^n$ if V is affine, or from $\mathbb{P}^n$ if V is projective.

Definition 1.2.5. The *dimension* of an algebraic variety V, denoted by $\dim(V)$, is defined as the largest integer n such that there exists a chain of distinct subvarieties of V, $V_0 \subsetneq V_1 \subsetneq ... \subsetneq V_n = V$. Algebraic varieties of dimension 1 are called *curves*, and those of dimension 2 are called *surfaces*. Both $\mathbb{A}^n$ and $\mathbb{P}^n$ are varieties of dimension n.

If W is a subvariety of V, we define the *codimension* of W as $\operatorname{codim}(W) := \dim(V) - \dim(W)$. Subvarieties of codimension 1 are called *divisors* (see section 1.2.4).

Definition 1.2.6. Let $V \subset \mathbb{A}^n$ be an affine algebraic variety. We define its *coordinate ring* as the quotient:

$$A(V) := \frac{\overline{k}[x_1, \cdots, x_n]}{I(V)}.$$

Since $I(V)$ is a prime ideal, we know that $A(V)$ is a domain. For each $P \in V$, let $M_P(V) \subset A(V)$ be the ideal of polynomials vanishing at P:

$$M_P(V) = \{f \in A(V) \mid f(P) = 0\}.$$

This is a maximal ideal of $A(V)$, and through localization we get a local ring $A(V)_{M_P(V)}$ with maximal ideal $\mathfrak{m}_P(V)$.

We can find copies of $\mathbb{A}^n$ inside of $\mathbb{P}^n$. Indeed, for each x_i, $0 \le i \le n$, the open set $U_i := \mathbb{P}^n \backslash V(x_i)$ can be identified with $\mathbb{A}^n$ by relating affine coordinates $(y_1, \cdots, y_n)$ to homogeneous coordinates $[y_0 : \cdots : y_i : 1 : y_{i+1} : \cdots : y_n]$.

If $V \subset \mathbb{P}^n$ is a projective variety, then we define $V_i := V \cap U_i$. For each $P \in V$ we define $\mathfrak{m}_P(V) = \mathfrak{m}_P(V_i)$, as long as $P \notin V(x_i)$.

Definition 1.2.7. Let V be an affine variety of dimension n and $P \in V$. Then, we say that P is *simple* or *non-singular* if:

$$dim_{\overline{k}}\left(\frac{\mathfrak{m}_P(V)}{\mathfrak{m}_P(V)^2}\right) = n.$$

Otherwise, we say that P is a *singular* point of V. If V has no singular points, we say that V is *smooth*.

1.2.2 Maps between Varieties

The first step towards defining morphisms between varieties is the definition of regular functions.

Definition 1.2.8. Let $Y \subset \mathbb{A}^n$ be an affine variety. A function $f : Y \to \overline{k}$ is *regular at a point* $P \in Y$ if there is an open neighbourhood U with $P \in U$ and $g, h \in \overline{k}[x_1, \cdots, x_n]$ such that $h \neq 0$ and $f = g/h$ on U. We say that f is regular on an open set $U \subset Y$ if it is regular for all $P \in U$.

For a projective variety $Y \subset \mathbb{P}^n$, we say that $f : Y \to \overline{k}$ is regular at P if there is a neighbourhood U of P and $g, h \in \overline{k}$ homogeneous polynomials of the same degree such that $h \neq 0$ and $f = g/h$ on U, and f is regular on $U \subset Y$ if it is regular for all $P \in U$.

Definition 1.2.9. We denote the ring of all regular functions on an open set U of a variety Y by $\mathcal{O}(U)$. Given $P \in Y$, we define the *local ring of P on Y*, $\mathcal{O}_P(Y)$, as the direct limit of $\mathcal{O}(U)$ for all neighbourhoods U of P:

$$\mathcal{O}_P(Y) := \varinjlim_{U \ni P} \mathcal{O}(U).$$

We define the *function field of* Y as the quotient field of any local ring $\mathcal{O}_P(Y)$:

$$\overline{k}(Y) := \mathrm{Quot}(\mathcal{O}_P(Y)).$$

We can also see $\overline{k}(Y)$ as the union $\bigcup_{P \in Y} \mathcal{O}_P(Y)$ of all local rings in Y.

A regular function f is defined over k if it is invariant under Galois action, that is, if $\sigma(f) = f$ for all $\sigma \in G$. Then we define $\mathcal{O}(U)_k, \mathcal{O}_P(Y)_k, k(Y)$ by taking only functions defined over k from $\mathcal{O}(U), \mathcal{O}_P(Y), \overline{k}(Y)$.

Definition 1.2.10. If X, Y are two varieties, a *morphism* between X and Y is a continuous map $\varphi : X \to Y$ such that if f is a regular function on $V \subset Y$, then $\varphi^*(f) = f \circ \varphi$ is regular on $U = \varphi^{-1}(V) \subset X$, that is, the map $\varphi^* : \mathcal{O}(V) \to \mathcal{O}(U)$ is a well-defined homomorphism.

A morphism is defined over k if it is invariant under Galois action.

We say that φ is an *isomorphism* if there is an inverse morphism $\psi : Y \to X$ such that $\varphi \circ \psi = id_Y$ and $\psi \circ \varphi = id_X$. If there is an isomorphism between two varieties X and Y, we say that they are isomorphic.

Definition 1.2.11. Let X and Y be two varieties, and U, V two open sets of X with morphisms $\varphi_U : U \to Y$ and $\varphi_V : V \to Y$. We say that φ_U and φ_V are equivalent if they agree on the intersection $U \cap V$. An equivalence class of pairs (U, φ_U), denoted by $\varphi : X \dashrightarrow Y$, is called a *rational map* from X to Y. We say that φ is dominant if $\varphi_U(U)$ is a dense subset of Y for some $U \subset X$. Notice that a rational map may not be defined on the entire variety X, but rather on an open set $U \subset X$.

A rational map is defined over k if it is invariant under Galois action, and it is said to be a *birational map* if it admits an inverse, that is, a rational map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = id_Y$ and $\psi \circ \varphi = id_X$ as rational maps. If there is a birational map between two varieties X and Y, we say that they are *birational* or *birationally equivalent*, and if it is defined over k, then we say they are *k-birationally equivalent.*

When a variety V of dimension n is birational to $\mathbb{P}^n$, we say that V is *rational* or *birationally trivial*, and if the rational map is defined over k, we say that it is *k-rational* or *k-birationally trivial.*

1.2.3 Blow-up

Definition 1.2.12. Let X be a variety of dimension n and $P \in X$ a point of X. Then, the *blow-up of X over P* is a variety $\tilde{X}$ endowed with a morphism $\varepsilon : \tilde{X} \to X$ such that:

i) $E = \varepsilon^{-1}(P)$ is a subvariety of $\tilde{X}$ isomorphic to $\mathbb{P}^{n-1}$, called the exceptional divisor of ε;

ii) ε gives an isomorphism when restricted to $\tilde{X}\backslash E$ and $X\backslash\{P\}$.

Example 1.2.13. (Blow-up of $\mathbb{P}^n$)

Let O be the point $[1:0:\cdots:0]$ in $\mathbb{P}^n$. We will construct the blow-up of $\mathbb{P}^n$ over O. Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be the rational map defined by:

$$\begin{aligned} \varphi\colon \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-1} \\ [x_0 : x_1 : \cdots : x_n] &\mapsto [x_1 : \cdots : x_n]. \end{aligned}$$

Notice that φ is defined everywhere except on O. Now consider the graph of this map, $\Gamma_\varphi = \{(P, \varphi(P);\ P \neq O\} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. The closure of Γ_φ in $\mathbb{P}^n$ is a variety $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the equations $x_i y_j - x_j y_i$, for $1 \leq i, j \leq n$, $i \neq j$. The natural projection to the first coordinate endows B with a morphism $\varepsilon : B \to \mathbb{P}^n$ defining the blow-up of $\mathbb{P}^n$ over O.

Definition 1.2.14. Let $\varepsilon : \tilde{X} \to X$ be the blow-up of X at a point P and E the exceptional divisor. Let $Y \subset X$ be a subvariety of X, then we define $\tilde{Y}$ as the closure of $\varepsilon^{-1}(Y)\backslash E$ in $\tilde{X}$, called the *strict transform of* Y.

1.2.4 Divisors

Definition 1.2.15. Let X be a smooth algebraic variety over k. We define the group of *divisors* of X as the formal sums over all the subvarieties of X of codimension 1.

$$\mathrm{Div}(X) := \left\{ \sum_C n_C C \;\middle|\; \begin{array}{c} C \subset X,\ codim(C) = 1, n_C \in \mathbb{Z}, \\ n_C = 0 \text{ for all but finitely many } C \end{array} \right\}.$$

Together with the natural addition and multiplication, the divisors of X have the structure of a free $\mathbb{Z}$-module. We can define the group of divisors of X over k, denoted by $\mathrm{Div}(X)_k$, as a subgroup of $\mathrm{Div}(X)$. We say that a divisor $D = \sum n_i C_i \in \mathrm{Div}(X)$ is k-rational, that is, $D \in \mathrm{Div}(X)_k$ if $n_i = n_j$ whenever $\sigma(C_i) = C_j$ for some $\sigma \in \mathrm{Gal}(\overline{k}/k)$.

Definition 1.2.16. A divisor is called a *prime divisor* if the index $n_C = 1$ occurs for a single irreducible subvariety, and equal is to zero for all others.

An *effective divisor* is a divisor $D = \sum n_C C$ such that $n_C \geq 0$ for all subvarieties C. We write $D \geq 0$ when D is effective.

Definition 1.2.17. The *support* of a divisor $D = \sum n_C C$ is defined as the union of all irreducible subvarieties C such that $n_C \neq 0$. We denote this by $\mathrm{Supp}\, D = \cup_{n_C \neq 0} C$.

The *degree* of a divisor D is the sum of the degrees of every C multiplied by the coefficient n_C, where $\deg(C) = \deg(f)$ for $C = V(f)$. We write $\deg(D) = \sum_C n_C \cdot \deg(C)$, and the group of divisors of degree 0 is denoted by $\mathrm{Div}^0(X)$. We also define naturally $\mathrm{Div}^0(X)_k := \mathrm{Div}^0(X) \cap \mathrm{Div}(X)_k$.

Given a non-zero rational function $f \in \overline{k}(X)$, we can associate to it a divisor $\operatorname{div} f$. Given a prime divisor C, f can be written as g/h, with $g, h \in \mathcal{O}_C(X)$. We define the *order of vanishing* of a function $\varphi \in \mathcal{O}_C(X)$ as $\operatorname{ord}_C(\varphi) := \operatorname{len}_{\mathcal{O}_C(X)}\left(\dfrac{\mathcal{O}_C(X)}{(\varphi)}\right)$, where $\operatorname{len}_R(M)$ is the length of the R-module M (see [Mat86, Sec. 2]). Thus, we can define the order of f along C as $\operatorname{ord}_C(f) := \operatorname{ord}_C(g) - \operatorname{ord}_C(h)$. With this, we define:

$$\operatorname{div} f = \sum_C \operatorname{ord}_C(f) C.$$

Remark 1.2.18. Here, the ring $\mathcal{O}_C(X)$ is the local ring of the generic point ε of the subvariety C (see [Har77, Ch. II, Ex. 2.3.4]).

Definition 1.2.19. A divisor D such that $D = \operatorname{div} f$ for some $f \in \overline{k}(X)$ is called a *principal divisor*. Two divisors $D_1, D_2 \in \operatorname{Div}(X)$ are called *linearly equivalent* if $D_1 - D_2 = \operatorname{div} f$ for some $f \in \overline{k}(X)$, denoted by $D_1 \sim D_2$.

The *Picard group* of X is defined by the quotient:

$$\operatorname{Pic}(X) := \operatorname{Div}(X)/\sim.$$

We also define the degree zero class group $J(X)$:

$$J(X) := \operatorname{Div}^0(X)/\sim.$$

We also say that two k-rational divisors are linearly equivalent over k if their difference is a principal divisor of a function $h \in k(C)$, thus defining naturally $\operatorname{Pic}(X)_k$ and $J(X)_k$.

Definition 1.2.20. For every variety X, we can define a class of divisors $K_X \in \operatorname{Pic}(X)$ through the differential forms in X, called the *canonical divisor of* X (see [Sha77, Ch. III, Sec. 3.6]).

Example 1.2.21. (Picard group of the Projective space)

If $X = \mathbb{P}^n$, then all principal divisors have degree 0. The converse is also true: if $D = n_{C_1}C_1 + ... + n_{C_k}C_k$, then for each $i = 1, ..., k$, there is a homogeneous polynomial p_i such that $p_i = 0$ defines C_i, and $\deg(p_i) = \deg(C_i)$. The rational function defined by $f = \prod_{i=1}^k p_i^{n_{C_i}}$ is such that $\operatorname{div} f = D$.

With this result, we know that two divisors D_1 and D_2 are equivalent in $\operatorname{Pic}(\mathbb{P}^n)$ if and only if $\deg(D_1) = \deg(D_2)$, therefore $\operatorname{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

A morphism $\varphi : X \to Y$ induces a group homomorphism $\varphi^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ by taking prime divisors C to the prime divisors in $\varphi^{-1}(C)$ with some multiplicity (see [Sha77, Ch. III, Sec. 1.2]). This is referred to as the *pullback map*. Similarly, one can define a *pushforward map* $\varphi_* : \operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ (see [Bea96, Sec. I.1]).

1.2.5 The Néron-Severi Group

If X and T are irreducible algebraic varieties, for each $t \in T$ there is a natural embedding $j_t : x \mapsto (x, t)$ of X into $X \times T$. Given a divisor $C \in \mathrm{Div}(X \times T)$ such that $\mathrm{Supp}\, C \not\supset X \times \{t\}$, the pullback by j_t gives us a divisor $j_t^*(C) \in \mathrm{Div}(X)$.

Definition 1.2.22. A map $f : T \to \mathrm{Div}(X)$ is called an *algebraic family of divisors* if there is a divisor $C \in \mathrm{Div}(X \times T)$ such that $j_t^*(C)$ is well defined and $f(t) = j_t^*(C)$ for every $t \in T$.

Two divisors $D_1, D_2 \in \mathrm{Div}(X)$ are called *algebraically equivalent* if they are in the same algebraic family, that is, if there is $f : T \to \mathrm{Div}(X)$ an algebraic family of divisors such that $f(t_1) = D_1$ and $f(t_2) = D_2$ for $t_1, t_2 \in T$. We denote this equivalence by $D_1 \approx D_2$.

See [Sha77, Ch. III, Sec. 4.4] for a proof that this is indeed an equivalence relation.

Lemma 1.2.23. *If $D_1 \sim D_2$, then $D_1 \approx D_2$. That is, linear equivalence implies algebraic equivalence.*

Proof. See [Sha77, Ch. III, Sec. 4.4.D]. □

Definition 1.2.24. The *Néron-Severi group* of X is defined by the quotient of the divisor group by algebraic equivalence:

$$\mathrm{NS}(X) := \mathrm{Div}(X)/\approx .$$

We know that the Néron-Severi group is finitely generated (see [Sha77, Ch. III, Thm 4.4.D]). Its rank is denoted by $\rho(X)$ and is called the *Picard number* of X. The elements of $\mathrm{NS}(X)$ defined over k are denoted by $\mathrm{NS}(X)_k$ and its rank is denoted by $\rho(X)_k$.

1.3 Algebraic Curves

1.3.1 Basic Properties

In this section we will state some classic results of the theory of algebraic curves.

Proposition 1.3.1. *Let $\varphi : C \to V$ be a rational map from a projective curve C to a projective variery V. If $P \in C$ is a non-singular point, then φ is regular at P. In particular, if C is smooth, then φ is a morphism.*

Proof. See [Ful89, Sec. 7.1]. □

Proposition 1.3.2. *Let $\varphi : C_1 \to C_2$ be a morphism between two curves. Then, φ is either constant or surjective.*

Proof. See [Har77, Ch. II, Prop. 6.8]. □

Definition 1.3.3. Let P be a point on a curve C. We define the *multiplicity of P at C* as:

$$m_P(C) := \lim_{n\to\infty} \dim_{\overline{k}} \left(\frac{\mathfrak{m}_P(C)^n}{\mathfrak{m}_P(C)^{n+1}} \right)$$

If P is a simple point of C, then $m_P(C) = 1$, and if P is singular, then $m_P(C) > 1$.

Definition 1.3.4. Let $C, D \subset S$ be two projective curves on a projective surface S and $P \in C \cap D$. If f, g are the functions that define C and D respectively on $\mathcal{O}_P(S)$, then, we define the *intersection number of C and D at P* as:

$$I(P, C \cap D) := \dim_{\overline{k}} \left(\frac{\mathcal{O}_P(S)}{(f, g)} \right).$$

The intersection number describes how C and D intersect at a point P. For example, if $P \notin C \cap D$, then $I(P, C \cap D) = 0$, if P is a simple point of C and D and they intersect transversely at P, then $I(P, C \cap D) = 1$.

Theorem 1.3.5 (Bézout's Theorem)**.** *Let $C = V(f)$ and $D = V(g)$ be two projective plane curves over an algebraically closed field such that f and g have no common factor,* $\deg(f) = m$ *and* $\deg(g) = n$*. Then*

$$\sum_{P \in \mathbb{P}^2} I(P, C \cap D) = mn.$$

Proof. See [Ful89, Sec. 5.3]. □

Notice that the conditions are necessary: the intersection of two curves over a field k may not be defined over k. A simple example is given by looking at the curves $V(x^2 + y^2 - z^2)$ and $V(x - 2z)$ over $\mathbb{R}$. Over $\mathbb{C}$, they have two complex intersection points $[2 : i\sqrt{3} : 1]$ and $[2 : -i\sqrt{3} : 1]$. At the same time, affine curves also may not intersect, for example, the parallel lines $V(x)$ and $V(x - 1)$ do not intersect in $\mathbb{A}^2_k$ for any field k.

1.3.2 Divisors over Curves and the Riemann-Roch Theorem

Let C be a smooth algebraic curve over k. The group of divisors $\mathrm{Div}(C)$ will then be the formal sums of points of C. Given $f \in \overline{k}(C)$ with zeroes of order n_i in points P_i and poles of order n_j in points P_j, we know that

$$\operatorname{div} f = n_{i_1} P_{i_1} + \ldots + n_{i_k} P_{i_k} - n_{j_1} P_{j_1} - \ldots - n_{j_l} P_{j_l}.$$

Given a divisor $D \in \text{Div}(C)$, we define $L(D)$ to be the space of rational functions $f \in \overline{k}(C)$ such that $\text{div}\, f + D$ is effective, that is:

$$L(D) = \{f \in \overline{k}(C);\ \text{div}\, f + D \geq 0\}.$$

$L(D)$ forms a finite dimensional vector space over $\overline{k}(C)$, and its dimension is denoted by $l(D)$.

Theorem 1.3.6 (Riemann's Theorem). *Let C be a smooth curve. Then there exists a non-negative integer $g(C)$, called the geometric genus of C, such that for all $D \in \text{Div}(C)$, we have an inequality:*

$$l(D) \geq \deg(D) + 1 - g(C).$$

Proof. See [Ful89, Sec. 8.3]. □

Using the canonical divisor K_C, we can find a correcting term in the last theorem's inequality.

Theorem 1.3.7 (Riemann-Roch). *With the same hypothesis as before, we now have an equality:*

$$l(D) - l(K_C - D) = \deg(D) + 1 - g(C).$$

Proof. See [Ful89, Sec. 8.5]. □

When C is a smooth plane curve, we can compute $g(C)$ by the following formula:

Proposition 1.3.8 (Genus-degree formula). *Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Then, we have:*

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Proof. See [Ful89, Sec. 8.3, Prop. 5]. □

1.3.3 Elliptic Curves

Definition 1.3.9. An *elliptic curve* over a field k is a smooth projective genus 1 algebraic curve defined over k with at least one k-rational point O.

Throughout this text, we will reserve the letter E for elliptic curves. An elliptic curve over k will be denoted by E/k or simply by E when there is no confusion about the field over which it is defined, and the set of k-rational points of the curve is denoted by $E(k)$.

Every elliptic curve E/k can be described as a plane cubic, known as the *generalized Weierstrass form*, where the $a_i \in k$.

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2 + a_4xz^2 + a_6z^3;$$

If $char(k) \neq 2, 3$, then we can use a simpler equation, the *Weierstrass normal form,* with p and q in k:

$$y^2z = x^3 + pxz^2 + qz^3.$$

For an arbitrary equation in Weierstrass normal form to describe an elliptic curve, we need the associated cubic to be smooth ($O = [0 : 1 : 0]$ is always a k-rational point). This happens if and only if the *discriminant* Δ is different from 0:

$$\Delta = -16(4p^3 - 27q^2) \neq 0.$$

Lemma 1.3.10. *The set $E(k)$ can be endowed with a group structure.*

Proof. Given a point $P \in E(k)$, we will denote by $[P]$ the divisor $1 \cdot P \in \mathrm{Div}(E)$. Then, we can construct a bijection:

$$\alpha \colon E(k) \to J(E)_k$$
$$P \mapsto [P] - [O]$$

which endows $E(k)$ with the group structure of $J(E)_k$. □

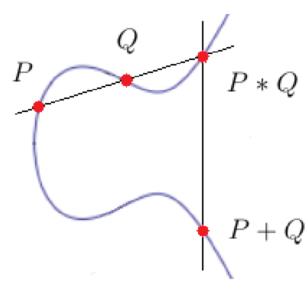


Figure 3: Geometric representation of the sum of two points on an elliptic curve E.

The group strucure of $E(k)$ has a simple geometric description. By Bézout's theorem, we know that every line L intersects a plane elliptic curve in 3 points counted with multiplicities. Then, given two points $P, Q \in E(k)$, we define $P * Q \in E(k)$ as the third point in the intersection of E and the line passing through P and Q (we take the tangent line if $P = Q$).

The sum of P and Q in the group $E(k)$ can be described as $P + Q = (P * Q) * O$.

Theorem 1.3.11. *(Mordell-Weil Theorem)*

Let k be a number field and E/k an elliptic curve. Then the group $E(k)$ is finitely generated:

$$E(k) \cong \mathbb{Z}^r \oplus E(k)_{tors}.$$

The number $r \in \mathbb{N}$ is called the rank of E/k, and is denoted by $rk(E(k))$. The finite group $E(k)_{tors}$ is the torsion part of $E(k)$, that is, elements $P \in E(K)$ such that $nP = 0$ for some $n \in \mathbb{Z}$.

Proof. See [Sil09, Ch. VIII, Sec. 1-6]. □

1.3.4 Pencils of Curves

Definition 1.3.12. A *pencil of curves of degree d* over $\mathbb{P}^2$ is a family of curves:

$$\Lambda : \{\lambda F(x,y,z) + \mu G(x,y,z) = 0 | [\lambda : \mu] \in \mathbb{P}^1\},$$

where $F(x,y,z)$ and $G(x,y,z)$ are degree d homogeneous polynomials that determine curves F and G without common components on $\mathbb{P}^2$.

If F and G have no common components, then every curve in Λ passes through the points in $F \cap G$. These are called the *base points* of Λ.

Example 1.3.13. Let $F(x,y,z) = x^3 + y^3 + z^3$ and $G(x,y,z) = xyz$. These curves generate a *pencil of cubics* called the *Hesse pencil*.

$$\mathcal{H} = \{\lambda(x^3 + y^3 + z^3) + \mu xyz = 0 | [\lambda : \mu] \in \mathbb{P}^1\}.$$

The base points of the Hesse pencil are the intersection of the cubics $x^3+y^3+z^3 = 0$ and $xyz = 0$, given by the set:

$$M = \left\{\begin{matrix} [0:1:-1], & [0:\omega:-1], & [0:\omega^2:-1], \\ [1:0:-1], & [\omega:0:-1], & [\omega^2:0:-1], \\ [1:-1:0], & [\omega:-1:0], & [\omega^2:-1:0] \end{matrix}\right\}.$$

Here, ω is the cubic root of unit $\frac{-1+\sqrt{-3}}{2}$.

In what follows, we see an important property of pencils of cubic curves.

Theorem 1.3.14 (Cayley-Bacharach)**.** *Let C_1 and C_2 be two different, possibly reducible, plane cubics such that C_1 and C_2 intersect in 9 different points. Then, every cubic that passes through 8 of these points will also pass through the ninth point. In other words, 8 points on $\mathbb{P}^2$ define a unique pencil of cubics if no 4 of them lie on a line and no 7 of them lie on a conic.*

Proof. See [Har77, Ch. V, Cor. 4.5]. □

1.4 Algebraic Surfaces

1.4.1 Divisors on Surfaces

Let S be a smooth projective surface over $\overline{k}$. In order to understand the geometry of S, we must understand the curves inside S.

Theorem 1.4.1. *There exists a symmetric bilinear product* $(\,\cdot\,) : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ *such that, if* C, D *are different irreducible curves on* S*, then:*

$$(C \cdot D) = \sum_{P \in S} I(P, C \cap D).$$

Proof. See [Bea96, Thm. I.4]. □

Remark 1.4.2. We may denote the product $(C \cdot D)$ as simply $C \cdot D$. In the case that $C = D$, we write $C \cdot C$ as C^2.

Definition 1.4.3. Let S be an algebraic surface and $D_1, D_2 \in \mathrm{Div}(S)$. Then we say that D_1 and D_2 are *numerically equivalent* if $D_1 \cdot D = D_2 \cdot D$ for all $D \in \mathrm{Pic}(S)$ (here, we consider the classes of D_1 and D_2 on $\mathrm{Pic}(S)$). We denote numerical equivalence by $D_1 \equiv D_2$.

Lemma 1.4.4. *If* $D_1 \approx D_2$*, then* $D_1 \equiv D_2$*. That is, algebraic equivalence implies numerical equivalence.*

Proof. See [SS17, Lem. 4.1]. □

Corollary 1.4.5. The product $(\,\cdot\,)$ can also be viewed as a symmetric bilinear pairing on $\mathrm{NS}(S)$.

Remark 1.4.6. The pairing $(\,\cdot\,)$ does not necessarily give a lattice structure to $\mathrm{NS}(S)$. For this to happen, $\mathrm{NS}(S)$ must be torsion free, otherwise $(\,\cdot\,)$ will be degenerate. On the following chapter we will see that when S is an elliptic surface, $\mathrm{NS}(S)$ will indeed have a lattice structure (see Section 2.2.1).

Theorem 1.4.7. *(Adjunction Formula)*

Let C *be a smooth curve inside a surface* S *with canonical divisor* K_S*. Then the following formula is true:*

$$2g(C) - 2 = K_S \cdot C + C^2.$$

Proof. See [Har77, Ch. I Prop. 1.5]. □

Definition 1.4.8. Let S be a projective surface. We call the *degree* of S the self-intersection of its canonical divisor K_S. It is denoted by $d_S := K_S^2$.

Example 1.4.9. Let $S = \mathbb{P}^2$. Then, Bézout's theorem (1.3.5) gives us an easy formula for calculating the product $D_1 \cdot D_2$. The canonical divisor will be equal to $K_S = -3H$, where H is the class of a line on $\mathrm{Pic}(S)$, and therefore, $d_S = 9H^2 = 9$.

1.4.2 Blow-up of a Surface

Lets now look at how the Picard group of a surface is changed through a blow-up.

Theorem 1.4.10. *Let S be a surface and $\varepsilon : \tilde{S} \to S$ the blow-up of a point $P \in S$ and E the exceptional curve. If C is a curve passing through P with multiplicity m, then:*

i) the pullback map gives us $\varepsilon^*(C) = \tilde{C} + mE$;

ii) let D, D' be divisors on S, then $\varepsilon^*(D) \cdot \varepsilon^*(D') = D \cdot D'$, $E \cdot \varepsilon^*(D) = 0$ and $E^2 = -1$;

iii) there is an isomorphism $\mathrm{Pic}(\tilde{S}) \cong \mathrm{Pic}(S) \oplus \mathbb{Z}$ defined by the map $(D, n) \mapsto D + nE$;

iv) similarly, $\mathrm{NS}(\tilde{S}) \cong \mathrm{NS}(S) \oplus \mathbb{Z}$;

v) the canonical divisor of $\tilde{S}$ can be written as $K_{\tilde{S}} = \varepsilon^*(K_S) + E$, so $d_{\tilde{S}} = d_S - 1$.

Proof. See [Bea96, Lem. II.2, Prop. II.3]. □

The next theorem serves as a converse to item (*ii*) of the above theorem, giving us a criterion for when a curve is in fact the exceptional curve of some blow-up.

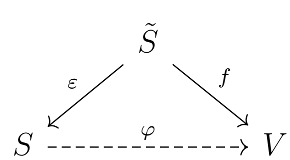
Theorem 1.4.11. *(Castenuovo's contractibility criterion)*

Let S be a projective surface and E a rational curve on S such that $E^2 = -1$. Then, there is a smooth surface S_0 and $P \in S_0$ such that S is isomorphic to the blow-up of S_0 on P, and E is the exceptional curve over P. In other words, every (-1)-curve over a surface S can be contracted to a smooth point P.

Proof. See [Bea96, Thm. II.17], or [Har77, Ch. V, Thm. 5.7]. □

Another great utility of the blow-up is the resolution of indeterminate points on rational maps.

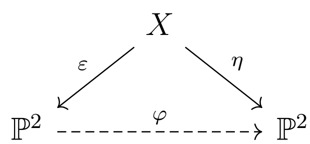
Theorem 1.4.12. *Let $\varphi : S \dashrightarrow V$ be a rational map from a surface S to a projective variety V. Then, there is a surface $\tilde{S}$ and a map $\varepsilon : \tilde{S} \to S$ given by a finite sequence of blow-ups and a morphism $f : \tilde{S} \to V$ such that $\varphi \circ \varepsilon = f$.*



Proof. See [Bea96, Thm II.7]. □

1.4.3 Cremona Transformations

A *Cremona transformation* in $\mathbb{P}^n$ is a birational map from $\mathbb{P}^n$ to itself. In this section we will see an example of a Cremona transformation in $\mathbb{P}^2$ and its action on plane curves. Let p_0, p_1, p_2 be points in $\mathbb{P}^2$ not all three in a line. Then we create a Cremona map $\varphi_{p_0,p_1,p_2} = \varphi$ by a blow-up ε and then a contraction η:



1) We blow up $\mathbb{P}^2$ in the points p_0, p_1, p_2. This takes us to a surface X with exceptional lines l_0, l_1, l_2 above each p_i. In this step, the self-intersection of every curve C decreases by $m_{p_0}(C) + m_{p_1}(C) + m_{p_2}(C)$, where $m_p(C)$ is the multiplicity of C at the point p. The lines l_{01}, l_{02}, l_{12}, where l_{ij} is the line through p_i and p_j, are blown up in two distinct points with multiplicity 1, so they become (-1)-curves.

2) We contract the lines l_{01}, l_{02}, l_{12}, going back to $\mathbb{P}^2$ with points p_{01}, p_{02}, p_{12} below them. After this step, the self-intersection of a curve $\tilde{C}$ in X will increase by $\tilde{C} \cdot \widetilde{l_{01}} + \tilde{C} \cdot \widetilde{l_{02}} + \tilde{C} \cdot \widetilde{l_{12}}$.

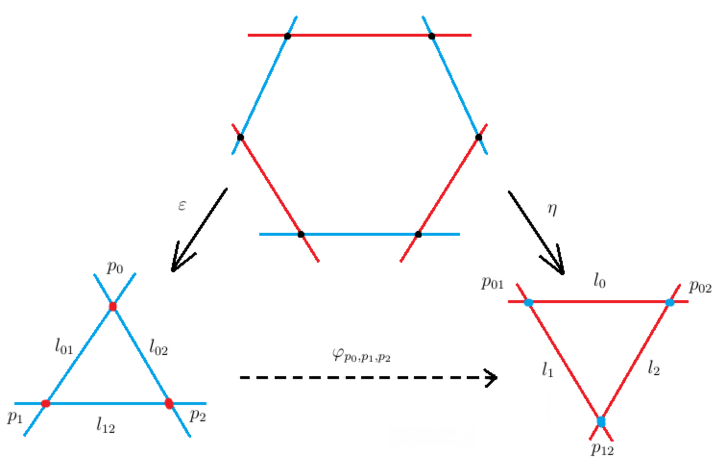


Figure 4: Illustration of a Cremona map.

We can describe how this Cremona transformation acts on curves in $\mathbb{P}^2$. If C is a curve of degree d passing through p_0, p_1, p_2 with multiplicities $\alpha_0, \alpha_1, \alpha_2$, then $\varphi_*(C)$ has degree $2d - \alpha_0 - \alpha_1 - \alpha_2$ and passes through p_{01}, p_{02}, p_{12} with multiplicities $(d - \alpha_0 - \alpha_1), (d - \alpha_0 - \alpha_2), (d - \alpha_1 - \alpha_2)$.

Example 1.4.13. The Cremona transformation of the points $P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0], P_3 = [0 : 0 : 1]$ can be shown explicitly by the rational map:

$$\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x : y : z] \mapsto [yz : xz : xy].$$

Lets see how φ acts on different plane curves:

C	$\deg(C)$	$m_{P_1}(C)$	$m_{P_2}(C)$	$m_{P_3}(C)$	$\varphi_*(C)$	$\deg(\varphi_*(C))$
$x = 0$	1	0	1	1	$P_1 = [1 : 0 : 0]$	0
$y = 0$	1	1	0	1	$P_2 = [0 : 1 : 0]$	0
$z = 0$	1	1	1	0	$P_3 = [0 : 0 : 1]$	0
$x = y$	1	0	0	1	$x = y$	1
$x + y = z$	1	0	0	0	$yz + xz = xy$	2
$3xz - xy = 2z^2$	2	1	1	0	$2xy + z^2 = 3yz$	2

1.4.4 Numerical Invariants

We can associate some birational invariants to every surface S using sheaf cohomology. These definitions play an important role in the classification of surfaces.

Definition 1.4.14. To every surface S we associate:

$$\begin{aligned} q(S) &= h^1(S, \mathcal{O}_S); \\ p_g(S) &= h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K_S)); \\ P_n(S) &= h^0(S, \mathcal{O}_S(nK_S)) \text{ for } n \geq 1; \\ \chi(S) &= h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S). \end{aligned}$$

Here, P_n are called the *plurigenera* of S, q is the *irregularity* of S, $p_g = P_1$ is the *geometric genus* and χ the *Euler characteristic* of S. The numbers $h^i(S, \mathcal{F})$ are the dimensions of the cohomology groups (see [Har77, Ch. III.2]) of the sheaf $\mathcal{F}$ (see [Har77, Ch. II.1]).

At last, we define the *Kodaira dimension*, another important invariant of surfaces.

Definition 1.4.15. Let S be a smooth, projective surface with canonical divisor K_S. We define the *Kodaira dimension* of S, denoted by $\kappa(S)$, as the smallest integer κ such that

$$\limsup_{n \to \infty} \frac{h^0(X, \mathcal{O}_S(K_S)^n)}{n^\kappa}$$

exists and is non-zero. If $h^0(X, \mathcal{O}_S(K_S)^n)$ vanishes for all positive integers n, we say that $\kappa(S) = -\infty$. Otherwise, we know that $0 \leq \kappa(S) \leq 2$.

See [Abr07] for more details on the definition of the Kodaira dimension.

Proposition 1.4.16. *The numbers q, p_g, P_n, χ and κ are invariant under birational transformations.*

Proof. See [Bea96, Prop. III.20]. □

1.4.5 Rational Surfaces

A surface S is called rational if it is birational to $\mathbb{P}^2$ over the field $\overline{k}$. Calculating the Kodaira dimension of $\mathbb{P}^2$, we know that every rational surface S has $\kappa(S) = -\infty$. Castelnuovo created a criterion to know if a surface is rational depending on its invariants.

Theorem 1.4.17. *(Castelnuovo's Rationality Criterion)*
Let S be a surface with $q(S) = P_2(S) = 0$. Then S is a rational surface.

Proof. See [Bea96, Thm. V.1]. □

Example 1.4.18. (del Pezzo surfaces, Conic Bundles and Hirzebruch surfaces)

1. A surface S is called a *del Pezzo* surface of degree d, with $1 \leq d \leq 9$, if it is isomorphic to the blow-up of $\mathbb{P}^2$ in $9 - d$ points in general position, that is, there are no 3 points on a line and neither 6 points on a conic. If $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then S is also called a del Pezzo surface of degree 8. Notice that if S is a del Pezzo of degree d, then $d_S = d$. Since blow-ups are *birational* morphisms, every del Pezzo is a rational surface.

2. A *conic bundle* is a surface S together with a surjective morphism $\varphi : S \to C$ such that for almost every $v \in C$, the fiber $\varphi^{-1}(v)$ is a conic curve. When the base curve of the conic bundle S is $\mathbb{P}^1$, S is a rational surface.

3. For each $n \geq 0$, we define a *Hirzebruch surface* $\mathbb{F}_n$ given by $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ (see [Bea96, Prop. III.15]). Every Hirzebruch surface can be constructed by a sequence of blow-ups and contractions of $\mathbb{P}^2$, for example, $\mathbb{F}_1$ is isomorphic to the blow-up of $\mathbb{P}^2$ over a single point p. When $n > 0$, the surfaces $\mathbb{F}_n$ always have a single curve C with negative self-intersection, and $C^2 = -n$.

Remark 1.4.19. Notice that a rational surface S may not be k-rational if there is no birational map $S \dashrightarrow \mathbb{P}^2$ that is defined over k.

1.4.6 Minimal Models

When dealing with a class of birational (over $\overline{k}$) varieties, it is useful to choose a simple variety inside your class to work with. These are often called the *minimal model* of your class.

Definition 1.4.20. A surface S defined over $\overline{k}$ is said to be a *minimal surface* if every birational morphism $S \to S'$, from S to another surface S', is an isomorphism.

We can also characterize minimal surfaces purely by its geometry: S is a minimal surface if and only if S contains no (-1)-curves. With this, we can easily see that $\mathbb{P}^2$ is a minimal model inside the class of rational surfaces.

Theorem 1.4.21. *For every projective surface S, there is a minimal surface S_0 such that there exists a birational morphism $S \to S_0$.*

Proof. See [Bea96, Prop. II.16]. □

Inside the class of rational surfaces, it is clear that $\mathbb{P}^2$ is a minimal surface. Then, a natural question arises: is $\mathbb{P}^2$ the only possible minimal surface is the class of rational surfaces? The next theorem shows that this is not true.

Theorem 1.4.22. *Let S be a minimal rational surface defined over $\overline{k}$. Then S is isomorphic to $\mathbb{P}^2$ or to one of the Hirzebruch surfaces $\mathbb{F}_n$ for $n \neq 1$.*

Proof. See [Bea96, Thm. V.10]. □

Now let S_1 and S_2 be surfaces defined over k. Notice that even if S_1 and S_2 are birational, they may not be k-birational if there is no birational map $\varphi : S_1 \dashrightarrow S_2$ defined over k. Therefore, the theory of minimal models changes when considering classes of k-birational surfaces. We define naturally:

Definition 1.4.23. A surface X defined over k is said to be *k-minimal* if every k-birational morphism $X \to X'$, where X' is a surface over k, is an isomorphism. Equivalently, X is a k-minimal surface if it does not contain a set of pairwise skew (-1)-curves that are invariant under the action of $\mathrm{Gal}(\overline{k}/k)$.

The next theorem, by Iskovskih [Isk80], classifies the minimal models of surfaces that are rational over $\overline{k}$.

Theorem 1.4.24. *Let X be a k-minimal rational surface. Then, X is isomorphic to a surface in one of the following families:*

1. *A del Pezzo surface with* $\mathrm{Pic}(X)_k = \mathbb{Z}$.
2. *A conic bundle with* $\mathrm{Pic}(X)_k = \mathbb{Z}^2$.

Proof. See [Isk80, Thm. 1]. □

Let S be a surface defined over k. Knowing a k-minimal model of a surface gives us a criterion for when S is k-rational.

Theorem 1.4.25. *Let X be a k-minimal rational surface of degree d_X. If $d_X \leq 4$, then X is not k-rational. If $d_X \geq 5$ and X has at least one point defined over k, then X is k-rational.*

Proof. See [Sal16, Thm. 2.7]. □

Chapter 2

Elliptic Surfaces

2.1 Elliptic Surfaces

2.1.1 Basic Definitions

Let S a smooth projective surface and C a smooth projective curve, both defined over a perfect field k. A surjective morphism $\pi : S \to C$ defined over k is called a *fibration* of S over the base curve C. For each point $v \in C(\bar{k})$, $F_v = \pi^{-1}(v)$ is called the *fiber* over v.

We call $\pi : S \to C$ a *genus 1 fibration* if all fibers F_v, except finitely many, are smooth curves of genus 1.

A *section* of $\pi : S \to C$ is a map $\sigma : C \to S$ such that $\pi \circ \sigma = id_C$. A fibration of S may admit many sections, and by convention, one of them is called the *zero-section* (σ_0). We denote the set of sections of S over C by $S(C)$. If a section σ is defined over k, we call it a k-section.

Definition 2.1.1. A smooth algebraic surface S is called an *elliptic surface* if:

i) S is endowed with a genus 1 fibration $\pi : S \to C$ with a section $\sigma_0 : C \to S$ defined over k;

ii) S is *relatively minimal* with respect to π, that is, no fiber contains (-1)-curves;

iii) there is at least one singular fiber.

If σ_0 is a k-section, then every fiber F_v over a point $v \in C(k)$ contains at least one k-rational point, $\sigma_0(v)$. Therefore, the fibers F_v are elliptic curves for all except finitely many $v \in C(k)$.

2.1.2 Geometry of Elliptic Surfaces

In what follows we study the geometry of elliptic surfaces. More precisely, we analyse the behaviour of its divisors.

Let $\Gamma \subset S$ be a curve inside S. Then, restricting the map π to Γ, we get a morphism between two curves $\pi|_\Gamma : \Gamma \to C$. By (1.3.2), we know that $\pi|_\Gamma$ is either constant or surjective. If $\pi(\Gamma) = \{v\}$, then Γ lies inside the fiber F_v, and we say that Γ is *vertical*. If $\pi(\Gamma) = C$, then we say that Γ is *horizontal*. This allows us to express every divisor as a sum of two divisors

$$D = D' + D''$$

with D' vertical (the components of $\operatorname{Supp} D'$ are vertical) and D'' horizontal (the components of $\operatorname{Supp} D''$ are horizontal). From the above discussion, we see that every fiber F_v is a vertical divisor, and the image $\sigma(C)$ of every section is an horizontal divisor.

Proposition 2.1.2. *Let F_v and $F_{v'}$ be any two different fibers of an elliptic surface $\pi : S \to C$. Then, F_v and $F_{v'}$ are equivalent inside $\mathrm{NS}(S)$. Furthermore, the class F of fibers has self intersection 0.*

Proof. The first affirmation follows directly from the definition of algebraic equivalence (1.2.22). To see that $F^2 = 0$, take two different fibers F_v and $F_{v'}$ in the class F. Then, $F \cdot F = F_v \cdot F_{v'}$, and by the description of the intersection product (1.4.1), $F_v \cdot F_{v'} = 0$, since they clearly do not intersect. □

Theorem 2.1.3. *The canonical divisor K_S is algebraically equivalent to $(2g(C) - 2 + \chi(S))F$. Consequently, the degree of the surface is $d_S = 0$.*

Proof. See [SS17, Thm. 5.28]. □

This gives us a way to calculate the self-intersection of sections inside S.

Proposition 2.1.4. *Let $D = \sigma(C)$ be the image of a section inside S. Then, $D^2 = -\chi(S)$ and $D \cdot F = 1$.*

Proof. See [SS17, Cor. 5.29]. □

2.1.3 The Generic Fiber and the Kodaira-Néron Model

There are two equivalent ways of looking at elliptic surfaces: for any given elliptic surface $\pi : S \to C$, the generic fiber of π, that is, the fiber over the generic point of C, is an elliptic curve E over the function field $K = k(C)$, with origin O corresponding to σ_0. The connection between S and E/K is described in the following theorem.

Theorem 2.1.5. *The set of sections $S(C)$, has a group structure. Furthermore, $S(C)$ is isomorphic to the group of points of the generic fiber, $E(K)$. This group is called the Mordell-Weil group of S.*

Proof. See [SS17, Prop. 5.4]. □

A very natural question arises, namely, given a curve C over k and an elliptic curve E/K, is there an elliptic surface with E as its generic fiber? The following definition and theorem answer that question.

Definition 2.1.6. An elliptic surface $\pi : S \to C$ such that E is the generic fiber of π is called a *Kodaira-Néron model* of E/K.

Theorem 2.1.7. *Every elliptic curve E over a function field K has a Kodaira-Néron model, and it is unique up to isomorphism.*

Proof. See [SS17, Thm. 5.17]. □

2.1.4 Singular Fibers

Definition 2.1.8. For an elliptic surface $\pi : S \to C$, we define the sets:

$$Sing(\pi) = \{v \in C;\ F_v \text{ is singular}\};$$
$$R = Red(\pi) = \{v \in C;\ F_v \text{ is reducible}\}.$$

Both $Sing(\pi)$ and $Red(\pi)$ are stable under the action of $G = Gal(\overline{k}/k)$. Every fiber is a divisor of S, and we can write every F_v as:

$$F_v = \sum_{i=0}^{m_v - 1} \mu_{v,i}\Theta_{v,i},$$

where m_v is the number of distinct irreducible components of F_v, $\Theta_{v,i}$ is an irreducible component of F_v for $0 \leq i \leq m_v - 1$ and $\mu_{v,i}$ is the multiplicity of $\Theta_{v,i}$ in F_v.

Theorem 2.1.9. *The following statements are true:*

i) Every F_v intersects the zero section (O) at a unique component, which we denote by $\Theta_{v,0}$, with coefficient $\mu_{v,0} = 1$.

ii) If F_v is singular and irreducible, then it is a nodal or cuspidal rational curve.

iii) If F_v is a reducible fiber, then every component $\Theta_{v,i}$ is a smooth rational curve such that $(\Theta_{v,i})^2 = -2$.

Proof. For (i), notice that $\sigma_0(C) \cap F_v$ is just a single point, so only one component of F_v can intersect O. Since $O \cdot F = 1$, we must have $\mu_{v,0} = 1$.

Affirmation (ii) follows from the adjunction formula for singular curves (See [BHPV04, Sec II.11]). The arithmetic genus of F_v is 1 and since F_v must have a singularity, $g(F_v) = 0$ and there is a point $P \in F_v$ such that $m_P(F_v) = 2$.

To see that (iii) is true, notice that $\Theta_{v,i} \cdot F_v = 0$, since if we take another fiber $F_{v'}$ in the class F, $\Theta_{v,i}$ and $F_{v'}$ do not intersect. Then, $\Theta_{v,i} \cdot (\sum_j \Theta_{v,j}) = \Theta_{v,i} \cdot (\sum_{j \neq i} \Theta_{v,j}) + \Theta_{v,i}^2 = 0$. Since F_v is connected, $\Theta_{v,i} \cdot (\sum_{j \neq i} \Theta_{v,j}) \geq 0$ and consequently, $\Theta_{v,i}^2 \leq 0$. By the adjunction formula (1.4.7), $2g(\Theta_{v,i}) - 2 = K_S \cdot \Theta_{v,i} + \Theta_{v,i}^2$. Since $K_S = (2g(C) - 2 + \chi(S))F$, we have $K_S \cdot \Theta_{v,i} = 0$, therefore, $g(\Theta_{v,i}) = 0$ and hence $\Theta_{v,i}^2 = -2$.

□

Definition 2.1.10. For each $v \in R$, we define the intersection matrix of the reducible fiber F_v:

$$A_v = ((\Theta_{v,i} \cdot \Theta_{v,j}))_{1 \leq i,j \leq m_v - 1}.$$

Theorem 2.1.11. *(Kodaira, Néron, Tate)*

Let F_v be a reducible singular fiber with m components. Then F_v must be equal to one of the following types:

- I_m: *$F_v = \Theta_0 + \cdots + \Theta_m - 1$ where, if $m \geq 3$, then $(\Theta_i \cdot \Theta_{i+1}) = 1$ for $0 \leq i \leq m-2$, and $(\Theta_{m-1} \cdot \Theta_0) = 1$. When $m = 2$, Θ_0 and Θ_1 intersect in 2 points transversally so that $(\Theta_0 \cdot \Theta_1) = 2$.*

- I_b^*: *$F_v = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + \cdots + 2\Theta_b + 4$, $m = b + 5$, $b \geq 0$, where $(\Theta_0 \cdot \Theta_4) = (\Theta_1 \cdot \Theta_4) = 1$, $(\Theta_2 \cdot \Theta_{b+4}) = (\Theta_2 \cdot \Theta_{b+4}) = 1$, and $(\Theta_4 \cdot \Theta_5) = \cdots = (\Theta_{b+3} \cdot \Theta_{b+4})$.*

- III: *$F_v = \Theta_0 + \Theta_1$, $m = 2$, where Θ_0 and Θ_1 intersect at a single point and $(\Theta_0 \cdot \Theta_1) = 2$.*

- IV: *$F_v = \Theta_0 + \Theta_1 + \Theta_2$, $m = 3$, where all components meet at a single point and $(\Theta_0 \cdot \Theta_1) = (\Theta_0 \cdot \Theta_2) = (\Theta_1 \cdot \Theta_2) = 1$.*

- II^*: *$F_v = \Theta_0 + 2\Theta_1 + 4\Theta_2 + 6\Theta_3 + 5\Theta_4 + 4\Theta_5 + 3\Theta_6 + 2\Theta_7 + 2\Theta_8$, $m = 9$, and $(\Theta_0 \cdot \Theta_7) = (\Theta_7 \cdot \Theta_6) = (\Theta_6 \cdot \Theta_5) = (\Theta_5 \cdot \Theta_4) = (\Theta_4 \cdot \Theta_3) = (\Theta_3 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_1) = (\Theta_3 \cdot \Theta_8) = 1$.*

- III^*: *$F_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6 + 2\Theta_7$, $m = 8$, where $(\Theta_0 \cdot \Theta_1) = (\Theta_1 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_3) = (\Theta_3 \cdot \Theta_4) = (\Theta_4 \cdot \Theta_5) = (\Theta_5 \cdot \Theta_6) = (\Theta_3 \cdot \Theta_7) = 1$.*

- IV^*: *$F_v = \Theta_0 + \Theta_1 + 2\Theta_2 + 3\Theta_3 + 2\Theta_4 + \Theta_5 + 2\Theta_6$, $m = 6$, where $(\Theta_1 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_3) = (\Theta_3 \cdot \Theta_4) = (\Theta_4 \cdot \Theta_5) = (\Theta_3 \cdot \Theta_6) = (\Theta_0 \cdot \Theta_6) = 1$.*

For $i \leq j$, if $(\Theta_i \cdot \Theta_j)$ is not given explicitly, then $(\Theta_i \cdot \Theta_j) = 0$.

Proof. See [Kod63, Thm. 6.2]. □

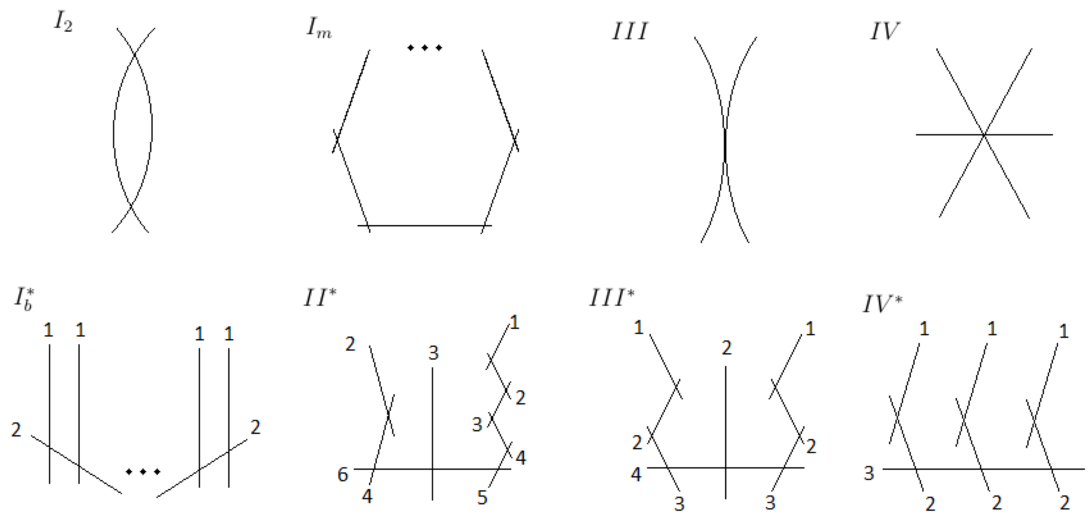


Figure 5: Possible singular fibers in the Kodaira Classification.

2.2 Mordell-Weil Lattices

Throughout the next subsections, we consider $\pi : S \to C$ an elliptic surface defined over an algebraically closed field $k = \overline{k}$.

2.2.1 The Néron-Severi Lattice

The intersection product $(D_1 \cdot D_2)$ gives the Néron-Severi group $\mathrm{NS}(S)$ a bilinear pairing $(\ \cdot\ ) : \mathrm{NS}(S) \times \mathrm{NS}(S) \to \mathbb{Z}$.

Theorem 2.2.1. *When S is an elliptic surface, algebraic and numerical equivalence are the same. That is, we can write $\mathrm{NS}(S) = \mathrm{Div}(S)/\equiv$.*

Proof. See [SS17, Thm 6.4]. □

As a corollary, we get that the intersection product is non-degenerate, giving $\mathrm{NS}(S)$ a lattice structure.

Definition 2.2.2. For S an elliptic surface with a section, the *trivial subgroup* or *trivial lattice* of S, $\mathrm{Triv}(S)$, is the subgroup of $\mathrm{NS}(S)$ generated by the zero-section and by the components of its fibers:

$$\mathrm{Triv}(S) = \langle (O), F \rangle \oplus \bigoplus_{v \in R} T_v$$

where $T_v = \langle \Theta_{v,i} \mid 1 \leq i \leq m_v - 1 \rangle$.

The correspondence between points of the generic fiber E and sections of S creates a way of relating $E(K)$ to the Néron-Severi group of S.

Theorem 2.2.3. *The map $P \mapsto (P)$ mod $Triv(S)$ gives us the isomorphism*

$$E(K) \cong \frac{\mathrm{NS}(S)}{\mathrm{Triv}(S)}.$$

Proof. See [SS17, Thm 6.5]. □

Essentially, the theorem above tells us that all of the horizontal divisors in $\mathrm{NS}(S)$ are given by sums of sections of $\pi : S \to C$.

Corollary 2.2.4. The group $E(K)$ is finitely generated. Therefore, this result can be view as a Mordell-Weil Theorem for elliptic curves over function fields. This result is known is greater generality as the Lang-Néron Theorem (see [Con06, Thm. 2.1]).

Corollary 2.2.5. (Shioda-Tate formula) Let $\pi : S \longrightarrow C$ be an elliptic surface and m_v the number of components of $\pi^{-1}(v)$, $v \in C$. Then:

$$r = \rho(S) - 2 - \sum_{v \in C} (m_v - 1).$$

Remark 2.2.6. We will use r to denote the rank $rk(E(K))$ of the generic fiber when we take $K = k(C)$ and k is algebraically closed. When working over a field k that is not algebraically closed, the $rk(E(k(C)))$ will be denoted by r_k.

2.2.2 The Height Pairing and the Mordell-Weil Lattice

Definition 2.2.7. The orthogonal complement of $\mathrm{Triv}(S)$ in $\mathrm{NS}(S)$ is called the essential sublattice of $\mathrm{NS}(S)$, $L(S) := \mathrm{Triv}(S)^{\perp}$.

Lemma 2.2.8. *For all $P \in E(K)$, there is an unique element of $\mathrm{NS}(S)_{\mathbb{Q}} = \mathrm{NS}(S) \otimes \mathbb{Q}$, denoted by $\varphi(P)$, such that:*

i) $\varphi(P) \cong (P)$ mod $\mathrm{Triv}(S)_{\mathbb{Q}}$;

ii) $\varphi(P) \perp \mathrm{Triv}(S)$.

The map $\varphi : E(K) \longrightarrow NS(S)_{\mathbb{Q}}$ is a group homomorphism and $ker(\varphi) = E(K)_{tors}$.

Proof. See [SS17, Lem. 6.16, 6.17]. □

Theorem 2.2.9. *The map φ induces an injection:*

$$\varphi^{'} : E(K)/E(K)_{tors} \longrightarrow L(S)_{\mathbb{Q}}$$

Now, for $P, Q \in E(K)$, we can define $\langle P, Q\rangle = -(\varphi(P)\cdot\varphi(Q))$. This induces the structure of a positive-definite lattice on $E(K)/E(K)_{tors}$.

Proof. See [SS17, Lem. 6.18, Thm. 6.20]. □

Definition 2.2.10. The pairing $\langle\cdot,\cdot\rangle$ is called the *height pairing*, and the lattice $(E(K)/E(K)_{tors}, \langle\cdot,\cdot\rangle)$ is called the *Mordell-Weil Lattice* of the elliptic curve E/K or of the elliptic surface $\pi : S \to C$.

Theorem 2.2.11. *(Explicit formula for the height pairing)*
For any $P, Q \in E(K)$, we have:

$$\langle P, Q\rangle = \chi(S) + (P\cdot O) + (Q\cdot O) - (P\cdot Q) - \textstyle\sum_{v\in R} contr_v(P,Q).$$

Here, $\chi(S)$ is the Euler characteristic of the surface S and $(P\cdot O), (Q\cdot O)$ and $(P\cdot Q)$ the intersection numbers between the sections $P, Q, O \in S(C)$, where O stands for the zero section. The number $contr_v(P,Q)$ stands for the local contribution at $v \in R$. Suppose that (P) intersects $\Theta_{v,i}$ and (Q) intersects $\Theta_{v,j}$, then:

$$contr_v(P,Q) = \begin{cases} (-A_v^{-1})_{i,j} & \text{if } i,j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $-(A_v^{-1})_{i,j}$ is the i,j entry of the inverse of the intersection matrix defined in (2.1.10).

Proof. See [SS17, Thm. 6.23]. □

2.2.3 The Narrow Mordell-Weil Lattice

Definition 2.2.12. Let $E(K)^0$ be the subgroup of $E(K)$ defined by:

$$E(K)^0 := \{P \in E(K);\ (P) \text{ meets } \Theta_{v,0} \text{ for all } v \in C(k)\}.$$

$E(K)^0$ is called the narrow Mordell-Weil group.

Theorem 2.2.13. *$E(K)^0$ is a torsion-free subgroup of $E(K)$.*

Proof. See [SS17, Thm. 6.42]. □

As a consequence, the height pairing gives $E(K)^0$ a lattice structure, and $(E(K)^0, \langle \cdot, \cdot \rangle)$ is called the *narrow Mordell-Weil lattice* of the elliptic curve E/K or of the elliptic surface $\pi : S \longrightarrow C$.

The lattice $E(K)^0$ connects the Mordell-Weil lattice to the essential lattice L.

Theorem 2.2.14. *The restriction of the map $\varphi : E(K) \longrightarrow \mathrm{NS}(S)_{\mathbb{Q}}$ to $E(K)^0$ is an isomorphism of lattices between $E(K)^0$ and the opposite essential lattice $L(S)^-$. Furthermore, φ injects $E(K)/E(K)_{tors}$ inside the dual lattice of L^-.*

$$\begin{array}{ccc} E(K)/E(K)_{tors} & \subset & (L^-)^\vee \\ \cup & & \cup \\ E(K)^0 & \cong & L^- \end{array}$$

If $\mathrm{NS}(S)$ *is unimodular, then we have* $E(K)/E(K)_{tors} \cong (L^-)^\vee$.

Proof. See [SS17, Thm. 6.45, 6.49]. □

2.2.4 Arithmetic of Lattices Associated to Elliptic Surfaces

So far, we have only seen the construction of the Mordell-Weil lattice for elliptic surfaces defined over algebraically closed fields.

Now, let k be a number field and C a curve defined over k. We write $K = k(C)$ and $K' = \overline{k}(C)$. Take an elliptic curve E/K with a Kodaira-Néron model $\pi : S \to C$. Then, the Galois group $G = \mathrm{Gal}(\overline{k}/k)$ acts on $E(K')$. The group $E(K)$ coincides with the subgroup of G-invariant points in $E(K')$. Thus, we can see $E(K)/E(K)_{tors}$ as a sublattice of the Mordell-Weil lattice of S. We call the rank r of $E(K')/E(K')_{tors}$ the *geometric rank* and the rank of $E(K)/E(K)_{tors}$, denoted by r_k, the *arithmetic rank*.

Theorem 2.2.15. *For any $P, Q \in E(K')$, $\sigma \in G$, we have that:*

$$\langle \sigma(P), \sigma(Q) \rangle = \langle P, Q \rangle.$$

Namely, the height pairing is stable under the action of G.

Proof. Notice that, by the formula (2.2.11), the pairing $\langle P, Q \rangle$ depends only on the surface S, the intersection products $(P \cdot O), (Q \cdot O), (P \cdot Q)$ and on $contr_v(P, Q)$ for each $v \in C$. By the description of the intersection product on (1.4.1), it is clear that $(\sigma(D_1) \cdot \sigma(D_2)) = (P_1 \cdot P_2)$ for any $D_1, D_2 \in \mathrm{NS}(S)$. Similarly, for $contr_v(P, Q)$, σ will take the reducible fiber F_v to another reducible fiber $F_{\sigma(v)}$ with the same intersection matrix A_v, so when we take the sum $\sum_v contr_v(P, Q)$ over all $v \in R$, the same contributions are counted. We conclude that the action of σ fixes the height pairing of any two points of $E(K')$. □

The Galois group G also acts on the fibers of $\pi : S \to C$. The set of reducible fibers in S is invariant under G, that is, the finite sum:

$$\mathcal{F} = \bigoplus_{v \in R} F_v$$

is stable under the action of G.

Taking invariant elements under the action of G on (2.2.5), we get an arithmetic version of the Shioda-Tate formula:

$$\rho_k = 2 + r_k + rk(\mathcal{F}^G). \tag{2.2.16}$$

Here, $\mathcal{F}^G$ denotes elements of $\mathcal{F}$ fixed by G, that is, all of the orbits of reducible fibers by the action of G.

In the Section 2.3.2, we give examples of the action of G on the reducible fibers of an elliptic surface.

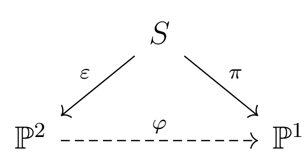
2.3 Rational Elliptic Surfaces

2.3.1 Basic Properties

Let k be a number field, and $F, G \in \overline{k}[x, y, z]$ two homogenous polynomials of degree 3 without common components. By Bézout's Theorem, the cubics defined by F and G meet at 9 points counted with multiplicity. We define a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1$:

$$\begin{aligned} \varphi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ P &\mapsto [F(P) : G(P)]. \end{aligned}$$

This map is not well defined exactly in the 9 points where F and G meet. Blowing up the points of indetermination, we obtain a rational elliptic surface S with an elliptic fibration $\pi : S \longrightarrow \mathbb{P}^1$.



When F and G intersect in 9 distinct points, we can write S as a surface in $\mathbb{P}^2 \times \mathbb{P}^1$ by the equation:

$$S : \lambda F(x, y, z) + \mu G(x, y, z) = 0.$$

The map π can be written explicitly as:

$$\pi : \; S \longrightarrow \mathbb{P}^1$$
$$([x:y:z],[\lambda:\mu]) \longmapsto [-\mu:\lambda].$$

Notice that, for each $P \in \mathbb{P}^2$, the surface S has a point $(P,[-G(P):F(P)])$, except when $P \in F \cap G$. In this case, S has a line above P, which is precisely the exceptional divisor of the blow-up.

Example 2.3.1. A classical example of a rational elliptic surface coming from a cubic pencil is the one constructed by the *Hesse pencil* (see Ex. 1.3.13). This surface, $H \subset \mathbb{P}^2 \times \mathbb{P}^1$, is given by the equation:

$$H : \lambda(x^3+y^3+z^3)+\mu xyz = 0.$$

The next result tells us that all rational elliptic surfaces can be created through this process over algebraically closed fields.

Theorem 2.3.2. *Over any algebraically closed field, all rational elliptic surfaces are isomorphic to the blow-up of $\mathbb{P}^2$ at the base points of a pencil of cubics.*

Proof. See [Mir89, Lem. IV.1.2] or [CD89, Thm. 5.6.1]. □

In particular, we have the following Lemma.

Lemma 2.3.3. *The following statements hold over an algebraically closed field:*

i) every rational elliptic surface S is fibered over $\mathbb{P}^1$ and the generic fiber E is defined over $K = k(\mathbb{P}^1)$;

ii) the Picard number of a rational elliptic surface over $\overline{k}$ is always equal to 10;

iii) the canonical divisor of a rational elliptic surface is $K_S = -F$;

iv) the sections $\sigma : \mathbb{P}^1 \to S$ are exactly the rational (-1)-curves inside S.

Proof. Statement (i) follows directly from Thm. 2.3.1, as S comes from the resolution of the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$.

For (ii), we use 2.3.1 combined with the properties of the blow-up of a surface (1.4.10.iv), since S is the blow-up of $\mathbb{P}^2$ over 9 points and each blow-up increases the Néron-Severi by 1.

By Prop. 2.1.3 and (i) above, we know that $K_S = (2g(\mathbb{P}^1) - 2 + \chi(S))F = (\chi(S)-2)F$. Since S is rational, $\chi(S) = \chi(\mathbb{P}^2) = 1$, and therefore $K_S = -F$, giving us statement (iii).

Affirmation (iv) comes from the adjunction formula (1.4.7): given a section $\sigma : \mathbb{P}^1 \to S$ with $C = \sigma(\mathbb{P}^1)$, we know that $g(C) = 0$, and therefore $C^2 = -K_S \cdot C - 2$.

Since C is a section and $K_S = -F$, we have by Prop. 2.1.4 that $C^2 = -1$. On the other hand, given C a rational (-1)-curve inside S, adjunction tells us that $K_S \cdot C = -C^2 - 2 = -2$, and therefore $F \cdot C = 1$. Then, since each fiber F_v intersects C in only one point, the inclusion map $\sigma : C \to S$ is a section of S.

□

Notice that, by item (ii) of the above Lemma, the Shioda-Tate formula (2.2.5) gives a direct correspondence between the Mordell-Weil rank of the generic fiber and the reducible fibers of the elliptic fibration:

$$r = 8 - \sum_{v \in C} (m_v - 1). \tag{2.3.4}$$

In what follows, we study the structure of the Mordell-Weil lattice of a rational elliptic surface $\pi : S \to \mathbb{P}^1$.

Theorem 2.3.5. *The Néron-Severi lattice* $\mathrm{NS}(S)$ *is unimodular.*

Proof. See [SS17, Prop. 7.5]. □

As a corollary, we know by Theorem 2.2.14 that:

$$E(K')/E(K')_{tors} \cong (L^-)^\vee.$$

Theorem 2.3.6. *If the group $E(K')$ has rank $r \geq 7$, then it is torsion free and the structure of the Mordell-Weil lattice is as follows:*

i) If $r = 8$, then π has no reducible fibers and:

$$E(K') = E(K')^0 \cong E_8.$$

ii) If $r = 7$, then π has one reducible fiber of type I_2 or III and:

$$\begin{array}{ccc} E(K') & \cong & E_7^\vee \\ \cup & & \cup \\ E(K')^0 & \cong & E_7 \end{array}$$

Proof. See [Shi90, Thm. 10.4]. □

Now, let X be a rational elliptic surface defined over k. The arithmetic version of the Shioda-Tate formula (2.2.16) gives us the relation:

$$\rho_k = 2 + r_k + rk(\mathcal{F}^G), \text{ with } \rho_k \leq 10. \tag{2.3.7}$$

Looking at X as surface over $\overline{k}$, we can evaluate the geometric rank $rk(E(K'))$ to find the structure of its Mordell-Weil lattice. Then, we can characterize $E(K)$ as a sublattice of $E(K')$.

Although X is a rational surface over $\overline{k}$, it is not necessarily k-rational (see Ex. 3.3.6). Thanks to Theorem 1.4.24 we can detect whether the surface is k-rational by inspecting its k-minimal model. The arithmetic version of the Shioda-Tate formula combined with (1.4.24) allows us to deduce the following first sufficient condition for a rational elliptic surface to be k-rational.

Theorem 2.3.8. *Let X be a rational elliptic surface defined over k. If $\rho(X)_k \geq 7$, then X is k-rational.*

Proof. Let X_0 be the k-minimal model of X. By Theorem 1.4.24, we know that $\rho(X_0)_k \leq 2$. Then, since each blow-up increases the Picard number by 1 by (1.4.10.iv), X must be the blow-up of X_0 over at least 5 different orbits of points by the action of G. Consequently, calculating the degree, we have $d_{X_0} \geq 5$ by 1.4.10.(v). By 1.4.25, we conclude that X and X_0 are k-rational. □

Notice that, on the other hand, not every rational elliptic surface that is k-rational has $\rho(X)_k \geq 7$ (see Ex. 3.2.13).

2.3.2 Examples of the Galois action on the fibers of a Rational Elliptic Surface

In what follows we study in four examples the action of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the fibers of a rational elliptic surface defined over $\mathbb{Q}$. Each has a different nature. In the first the reducible fibers are both defined over the ground field, while in the second the reducible fibers are not defined over the ground field, but conjugate under the action of G. The third example shows that G might fix a fiber, but still act on its different components. Finally, the fourth and last example shows both actions at the same time: G acts on conjugate reducible fibers and it's respective components.

Example 2.3.9. Let C_1 and C_2 be the plane cubics given by:

$$C_1: \; -x^3 - 9y^3 + 3x^2y + 7x^2z + 3xy^2 - 9y^2z - 6xz^2 - 18yz^2 - 6xyz = 0;$$
$$C_2: \; -x^3 - 4y^3 + 2x^2y + 9x^2z + 2xy^2 - 12y^2z - 8xz^2 - 16yz^2 - 4xyz = 0.$$

The blow-up of the 9 distinct base points of the cubic pencil generated by C_1 and C_2 gives us a rational elliptic surface $S_1 \subset \mathbb{P}^1$:

$$S_1 : \lambda C_1(x, y, z) + \mu C_2(x, y, z) = 0.$$

This surface has two reducible fibers of type I_2, above the points $v_1 = [0 : 1]$ and $v_2 = [1 : 0]$. In $\mathbb{P}^2$, the fibers F_{v_1} and F_{v_2} are given by C_1 and C_2. Each cubic can be factored in a conic and a line, given by the equations:

$$\begin{aligned}
Q_1 &: \ x^2 - 3y^3 - xz + 3yz = 0; \\
R_1 &: \ -x + 3y + 6z = 0; \\
Q_2 &: \ x^2 - 2y^3 - xz + 2yz = 0; \\
R_2 &: \ -x + 2y + 8z = 0.
\end{aligned}$$

Here, $C_1 = Q_1 \cdot R_1$ and $C_2 = Q_2 \cdot R_2$. By (2.3.4), since we have two fibers with two components, the Mordell-Weil rank of the surface S_1 must be 6 over $\overline{\mathbb{Q}}$. Over $\mathbb{Q}$, we can take Galois invariants on the arithmetic Shioda-Tate (2.3.7) to find:

$$\rho_{\mathbb{Q}} = r_{\mathbb{Q}} + 4.$$

Example 2.3.10. Take two cubics C_3 and C_4 given by:

$$\begin{aligned}
C_3 &: \ 2x^3 - 2x^2y + xy^2 - y^2z + yz^2 - xyz = 0; \\
C_4 &: \ -2x^2z + xy^2 - y^2z + 2xz^2 + yz^2 - xyz = 0.
\end{aligned}$$

The pencil defined by these cubics gives rise to a rational elliptic surface $S_2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ with the equation:

$$S_2 : \ \lambda C_3(x, y, z) + \mu C_4(x, y, z) = 0.$$

This surface has a pair of conjugate reducible fibers of type I_2 above the points $v = [-i : 1]$ and $\overline{v} = [i : 1]$. In $\mathbb{P}^2$, F_v is the union of a conic Q and a line L given by the equations:

$$\begin{aligned}
Q &: \ 2x^2 + (i+1)y^2 - 2xz - (i+1)yz = 0; \\
L &: \ x - iz = 0.
\end{aligned}$$

Similarly, $F_{\overline{v}}$ is the union the conjugates of Q and L by the action of the Galois group $\mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q})$, written $\overline{Q}$ and $\bar{L}$.

Since S_2 has two reducible fibers with two components, by (2.3.4), S_2 is a rational surface of Mordell-Weil rank 6 over $\overline{\mathbb{Q}}$. However, looking at S_2 over $\mathbb{Q}$, since F_v and $F_{\overline{v}}$ are not defined over $\mathbb{Q}$, we have only one orbit of fibers contributing to $\mathcal{F}^G$, and 2.3.7 gives us:

$$\rho_{\mathbb{Q}} = r_{\mathbb{Q}} + 3.$$

Example 2.3.11. Take the pencil defined by the cubics C_5 and C_6:

$$C_5: \; y^2z + yz^2 = x^3 - xz^2;$$
$$C_6: \; 6y^3 + 4xy^2 - 16y^2z + x^2y + 11yz^2 - 6xyz = 0.$$

Let S_3 be the rational elliptic surface defined by:

$$S_3: \; \lambda C_3(x,y,z) + \mu C_4(x,y,z) = 0.$$

The surface S_3 has Mordell-Weil rank 6 over $\overline{k}$, having a single reducible fiber of type I_3 above the point $v = [1:0]$. The fiber F_v is the union of three lines $L_1, L_2, \overline{L_2}$, given by:

$$L_1: y = 0;$$
$$L_2: x + (2 + i\sqrt{2})y - (3 + i\sqrt{2})z = 0;$$
$$\overline{L_2}: x + (2 - i\sqrt{2})y - (3 - i\sqrt{2})z = 0.$$

This fiber is fixed by G, however the components L_2 and $\overline{L_2}$ are conjugate by the action of $\mathrm{Gal}(\mathbb{Q}[i\sqrt{2}]/\mathbb{Q})$. Then, 2.3.7 gives us:

$$\rho_{\mathbb{Q}} = r_{\mathbb{Q}} + 3.$$

Example 2.3.12. Take the cubics C_7 and C_8:

$$C_7: \; 9x^3 - 2xy^2 + 5xz^2 = 0;$$
$$C_8: \; -z^3 + 3x^2z - 2y^2z = 0.$$

The surface S_4 is given by the blow-up of the fixed points of the cubic pencil generated by C_7 and C_8:

$$S_4: \; \lambda C_7 + \mu C_8 = 0.$$

Let σ, τ be the elements of $\mathrm{Gal}(\mathbb{Q}[\sqrt{2}, i]/\mathbb{Q})$ described by $\sigma: i \mapsto -i$ and $\tau: \sqrt{2} \mapsto -\sqrt{2}$. The surface S_4 has geometric Mordell-Weil rank 4, having two reducible fibers of type I_3 above $v = [-i:1]$ and $v^\sigma = [i:1]$ (see [Pas10, 2.4.3]). In $\mathbb{P}^2$, the fiber F_v is given by the line M defined over $\mathbb{Q}[i]$ and a pair of lines N, N^τ defined over $\mathbb{Q}[\sqrt{2}, i]$ and conjugate by τ.

$$M: \; x + iz = 0;$$
$$N: \; 3x - \sqrt{2}y - iz = 0.$$

The fiber F_{v^σ} is similarly given by the conjugates $M^\sigma, N^\sigma, N^{\sigma\tau}$. Looking at all the fiber components, we can see that there are two orbits, $\{M, M^\sigma\}$ and $\{N, N^\sigma, N^\tau, N^{\sigma\tau}\}$. Using 2.3.7, we get that:

$$\rho_{\mathbb{Q}} = r_{\mathbb{Q}} + 3.$$

2.3.3 Rational elliptic surfaces with geometric Mordell-Weil rank 7

Let S be a rational elliptic surface over $\overline{k}$ with $r = 7$. Then, by the Shioda-Tate formula (2.2.5), S admits only one reducible fiber, of type I_2 or III in the Kodaira classification. In [Fus06], Fusi describes all of the pencils of cubics that generate a rational elliptic surface with geometric Mordell-Weil rank 7. They can be one of the following:

i) a cubic pencil with the 6 base points over a conic Q and 3 over a line L transversal to Q;

ii) a cubic pencil with 8 base points over a cubic C with a node singularity, in which the node p_0 is one of the base points and all cubics pass through p_0 with a fixed direction;

iii) the same as (i) but L meets Q tangentially;

iv) the same as (iii) but the singular point is a cusp.

Above, there is no other dependence relation between the 9 points, that is, there is no other arrangement of 3 points on a line or 6 on a conic besides Q and L.

We also see in [Fus06] that we can use Cremona maps to show that (i) is an equivalent construction to (ii), and (iii) equivalent to (iv).

Definition 2.3.13. Over an algebraically closed field, we define the *construction* of a rational elliptic surface as a choice of cubic pencil Λ on $\mathbb{P}^2$. We say that two constructions Λ_1, Λ_2 are *equivalent* if there is a sequence of Cremona maps that take Λ_1 to Λ_2.

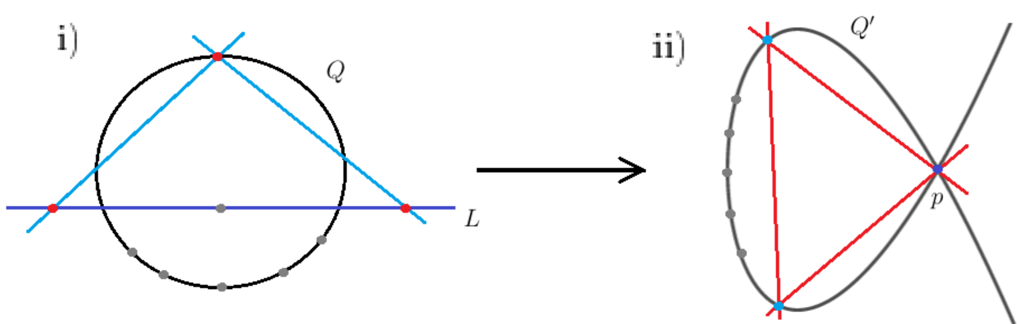


Figure 6: Cremona map showing the equivalence between constructions (i) and (ii).

If we apply a Cremona transformation to (i) in two base points of the line L and one point of the conic Q, then L is contracted to a point P and Q becomes a cubic with a node in P. The third point on the line becomes the fixed direction in the cubic pencil, giving us a cubic pencil of type (ii). In both these cases, the blow-up of the base points of the linear systems gives us a surface S with one reducible fiber of type I_2.

Similarly, applying a Cremona transformation to (iii) in two base points of L and one of Q, L is contracted to a point P and Q becomes a cubic with a cusp in P, matching the construction (iv). In these cases, the blow-up of the base points gives us a surface with one reducible fiber of type III.

Theorem 2.3.14. *Let S be a rational elliptic surface of Mordell-Weil rank 7. Then, over an algebraically closed field, S arises from a linear pencil of cubic curves on $\mathbb{P}^2$ as in (i) or (iii).*

Proof. See [Fus06, Thm. 2.7]. □

Chapter 3

Construction of Rational Elliptic Surfaces over $\mathbb{Q}$

3.1 Arithmetic of Rational Elliptic Surfaces with $r = 7$

In this chapter, we will follow Fusi's construction shown in the last Section, but now working over $\mathbb{Q}$. It is essential to emphasize that working over non-algebraically closed fields changes the theory drastically, as exemplified by the following Lemma.

Lemma 3.1.1. *Theorems 2.3.1 and 2.3.14 are not valid over* $\mathbb{Q}$.

Proof. This follows immediately from the fact that not every rational elliptic surface is $\mathbb{Q}$-rational (see Ex. 3.3.6). □

This makes it difficult to tell when two constructions are equivalent over $\mathbb{Q}$: notice that Def. 2.3.13 only works for surfaces arising from plane cubic pencils.

Let X be a rational elliptic surface defined over $\mathbb{Q}$ with geometric rank $r = 7$. By the Shioda-Tate formula (2.3.4), we know that X must have only one reducible fiber with two components, and since $\mathcal{F}$ is invariant under G, the fiber must be defined over $\mathbb{Q}$. Consequently, we have a direct correspondence between the G-invariant parts of ρ and r:

$$\rho_{\mathbb{Q}} = r_{\mathbb{Q}} + 3. \tag{3.1.2}$$

Using this formula along with Thm. 2.3.8, we know that X is $\mathbb{Q}$-rational if $r_{\mathbb{Q}} \geq 4$. This condition is not necessary (see Ex. 3.2.13).

Our aim is to study constructions of rational elliptic surfaces defined over $\mathbb{Q}$ with geometric Mordell-Weil rank 7 and show that given the reducible fiber type (i.e. fixed to be I_2 or III) there are different constructions of such surfaces that are non-equivalent. This is in contrast to the geometric case, where Fusi showed in

Thm. 2.3.14 that, over an algebraically closed field, once a rational elliptic surface with $r = 7$ and a type of reducible fiber are given, all constructions are equivalent.

Remark 3.1.3. In the case of surfaces with geometric Mordell-Weil rank 8, we still have examples of different constructions that are not $\mathbb{Q}$-equivalent (in the example 3.3.5, we will give an example of surface with $r = 8$ that is not $\mathbb{Q}$-rational). However, in here we choose to study the case with $r = 7$, since the presence of a reducible fiber creates a richer interplay between the geometry and the arithmetic of the rational elliptic surfaces.

3.2 Galois action on pencils of cubics over $\mathbb{Q}$

The simplest way of constructing a rational elliptic surface over $\mathbb{Q}$ with $r = 7$ is following the construction for an algebraically closed field. We do this by blowing-up $\mathbb{P}^2$ in the base points of a pencil of cubics Λ with one of the configurations described in [Fus06], such that Λ is generated by cubics defined over $\mathbb{Q}$. In this case, the base points of the pencil form a Galois-invariant set, and the blow-up $\varepsilon : S \to \mathbb{P}^2$ is defined over $\mathbb{Q}$. Therefore, every surface S created by this method is $\mathbb{Q}$-rational.

The greatest advantage of looking at rational elliptic surfaces arising from cubic pencils defined over $\mathbb{Q}$ is that it allows us to define an analogue to Def. 2.3.13.

Definition 3.2.1. Over $\mathbb{Q}$, we define the *construction* of a $\mathbb{Q}$-rational elliptic surface as a choice of cubic pencil Λ on $\mathbb{P}^2$, along with the structure of the Galois action on its base points. We say that two constructions Λ_1, Λ_2 are $\mathbb{Q}$-*equivalent* if there is a sequence of Cremona maps defined over $\mathbb{Q}$ that take Λ_1 to Λ_2. When the setting is clear, we may just say that the two constructions are equivalent.

Even in this setting, which is the most similar to the algebraically closed case, most constructions are not $\mathbb{Q}$-equivalent: the structure of the orbits of the G-action on the base points must be similar for this to happen. For example, even though the constructions (i) and (ii) in [Fus06] are equivalent over $\overline{\mathbb{Q}}$, they may not be $\mathbb{Q}$-equivalent. This will only happen if the Cremona map between them is defined over $\mathbb{Q}$.

Remark 3.2.2. Notice that we can compute $\rho_{\mathbb{Q}}$ based on the number of different orbits on the base points. For each orbit $\mathcal{O} = \{p_1, ..., p_n\}$, the divisor $E_1 + ... + E_n$, where E_i is the exceptional divisor above p_i, is defined over $\mathbb{Q}$, and increases the rank of the Néron-Severi group by 1. Consequently, we can evaluate the Mordell-Weil rank $r_{\mathbb{Q}}$.

We will classify the possible structures of these Galois orbits by their sizes in the different components of the reducible fiber. In the following tables, each line represents a possible configuration of the orbits on the base points by giving the quantities of n-orbits under $n\mathcal{O}$, that is, orbits with n elements. For example, an 1-orbit will be a single point fixed by G, so it will be a $\mathbb{Q}$-point.

The first table classifies the configurations on cubic pencils of type (i), with the first 6 columns giving the orbits on the points of the conic Q and the other 3 orbits on points of the line L. The second table classifies cubic pencils of type (ii), with the 7 columns showing the orbits over the cubic C. Notice that while there are 8 base points on C, we cannot have an 8-orbit, as the singular point must be fixed by G and is, therefore, defined over $\mathbb{Q}$.

Table 1: Case (i)

N	$\rho_{\mathbb{Q}}$	$1\mathcal{O}$	$2\mathcal{O}$	$3\mathcal{O}$	$4\mathcal{O}$	$5\mathcal{O}$	$6\mathcal{O}$	$1\mathcal{O}$	$2\mathcal{O}$	$3\mathcal{O}$
1	10	6						3		
2	9	6						1	1	
3	9	4	1					3		
4	8	6								1
5	8	4	1					1	1	
6	8	3		1				3		
7	8	2	2					3		
8	7	4	1							1
9	7	3		1				1	1	
10	7	2	2					1	1	
11	7	2			1			3		
12	7	1	1	1				3		
13	7		3					3		
14	6	3		1						1
15	6	2	2							1
16	6	2			1			1	1	
17	6	1	1	1				1	1	
18	6	1				1		3		
19	6		3					1	1	
20	6		1		1			3		
21	6			2				3		
22	5	2			1					1
23	5	1	1	1						1
24	5	1				1		1	1	
25	5		3							1
26	5		1		1			1	1	
27	5			2				1	1	
28	5						1	3		
29	4	1				1				1
30	4		1		1					1
31	4			2						1
32	4						1	1	1	
33	3						1			1

Table 2: Case (ii)

N	$\rho_{\mathbb{Q}}$	$1\mathcal{O}$	$2\mathcal{O}$	$3\mathcal{O}$	$4\mathcal{O}$	$5\mathcal{O}$	$6\mathcal{O}$	$7\mathcal{O}$
1	10	8						
2	9	6	1					
3	8	5		1				
4	8	4	2					
5	7	4			1			
6	7	3	1	1				
7	7	2	3					
8	6	3				1		
9	6	2	1		1			
10	6	2		2				
11	6	1	2	1				
12	5	2					1	
13	5	1	1			1		
14	5	1		1	1			
15	4	1						1

A configuration on the Nth line of the first table will be referred to as $(i).N$, while a configuration on the Nth line of the second table will be referred as $(ii).N$.

Although the cases (iii) and (iv) are never $\mathbb{Q}$-equivalent to (i) and (ii), since they are not even equivalent over $\overline{\mathbb{Q}}$, they have the same possible orbit structures, so we will omit its tables.

While these are all the possible structures of Galois orbits on the reducible fiber, the existence of a $\mathbb{Q}$-rational elliptic surface defined by the blow-up of these points is not guaranteed, as we need to ensure that these points are the base points of a pencil of cubics. For this to happen, we need to find an irreducible cubic C, a conic Q and a line L, all defined over $\mathbb{Q}$, such that the intersection points in $C \cap (Q \cup L)$ follow the structure of the above tables.

Remark 3.2.3. Notice that although rational elliptic surfaces that come from a pencil of cubics are always $\mathbb{Q}$-rational, not every $\mathbb{Q}$-rational elliptic surface arises from a pencil of cubics over $\mathbb{Q}$.

3.2.1 $\mathbb{Q}$-Equivalent Constructions

Recall from Def. 3.2.1 that for two different constructions to be $\mathbb{Q}$-equivalent, it is necessary for the Cremona map to be defined over $\mathbb{Q}$, that is, the map must come from 3 points that are closed under the action of G.

Proposition 3.2.4. *Constructions $(i).2$ and $(ii).2$ are equivalent:*

Proof. Let q be a point in Q and p, $\overline{p}$ be the pair of conjugate points in L, with $R, \overline{R}$ being the pair of conjugate lines between p, q and $\overline{p}, q$, respectively. The Cremona map φ defined by $q, p, \overline{p}$ will take $R, \overline{R}, L$ to fixed points $r, \overline{r}, l$ over $\varphi_* Q$, where $deg \varphi_* Q = 2 \cdot 2 - 1 = 3$, the points r and $\overline{r}$ are conjugate points over which $\varphi_* Q$ passes with multiplicity 1 and l is a node. Therefore, $\varphi_* Q$ will have 8 marked points (the 5 marked $\mathbb{Q}$-points from Q different than q, r, $\overline{r}$, and l), with 6 $\mathbb{Q}$- points and a pair of conjugates, fitting the construction $(ii).2$. □

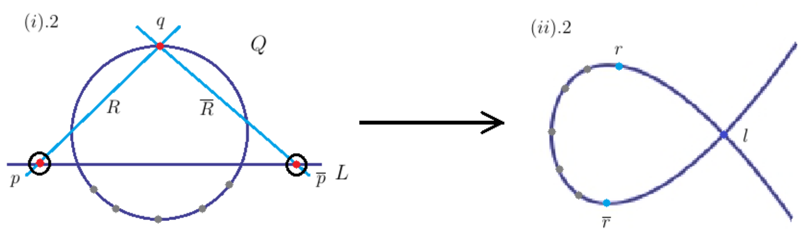


Figure 7: Cremona map over $\mathbb{Q}$ showing the equivalence between constructions (i).2 and (ii).2.

Proposition 3.2.5. *Constructions* $(i).2$ *and* $(i).3$ *are equivalent:*

Proof. Let q_1, q_2, q_3 be three $\mathbb{Q}$-points in Q, and R_{12}, R_{13}, R_{23} be the lines between them. Applying the Cremona map φ defined by q_1, q_2, q_3, we get $deg \varphi_* Q = 2 \cdot 2 - 3 = 1$, and $deg \varphi_* L = 1 \cdot 2 - 0 = 1$, so $\varphi_* Q$ will be a line with the 3 remaining $\mathbb{Q}$-points of Q and $\varphi_* L$ will be a conic with the points r_{12}, r_{13}, r_{23}, the $\mathbb{Q}$-point and the pair of conjugate points of L, fitting the construction $(i).3$. □

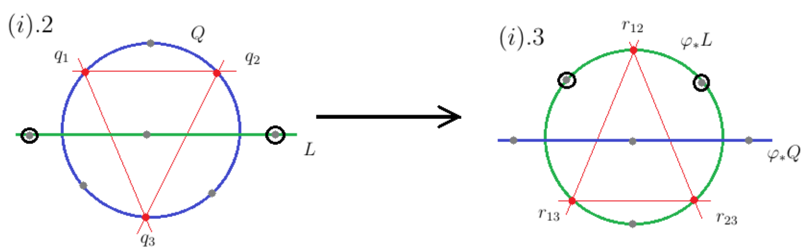


Figure 8: Cremona map over $\mathbb{Q}$ showing the equivalence between constructions (i).2 and (i).3.

Notice that $\rho_{\mathbb{Q}}$ and $r_{\mathbb{Q}}$ are invariants for equivalent constructions. This allows us to conclude that the configuration $(i).2$ cannot be $\mathbb{Q}$-equivalent to $(i).4$, as a cubic pencil in configuration $(i).2$ gives rise to an elliptic surface with $\rho_{\mathbb{Q}} = 9$ and $r_{\mathbb{Q}} = 6$, while $(i).4$ gives rise to surface with $\rho_{\mathbb{Q}} = 8$ and $r_{\mathbb{Q}} = 5$. On the other hand, it is possible that two different configurations with the same $\rho_{\mathbb{Q}}$ are not equivalent over k.

Proposition 3.2.6. *(ii).15 is not equivalent to any other configuration.*

Proof. We can see that there are no Cremona maps from this configuration defined over k: no set of three different base points is invariant under G. We can also look at possible candidates of equivalent configurations, namely the ones which give rise to surfaces with $\rho_k = 4$. Then, it is clear that none of them are equivalent to (ii).15, as they don't have a 7-orbit. □

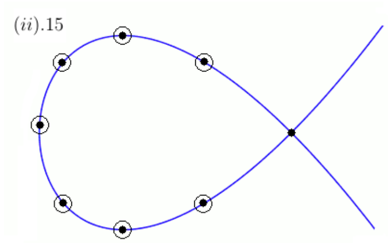


Figure 9: Base points in configuration (ii).15.

Using Cremona maps as in the propositions above, we can find all of the possible $\mathbb{Q}$-equivalences:

Proposition 3.2.7. *The configurations inside the same brackets below are $\mathbb{Q}$-equivalent.*

$$\rho_k = 10 : \{(i).1, (ii).1\};$$
$$\rho_k = 9 : \{(i).2, (i).3, (ii).2\};$$
$$\rho_k = 8 : \{(i).4, (i).6, (ii).3\}, \{(i).5, (i).7, (ii).8\};$$
$$\rho_k = 7 : \begin{Bmatrix} (i).8 & (i).9 \\ (i).12 & (ii).6 \end{Bmatrix}, \quad \{(i).10, (ii).7\}, \quad \{(i).11, (ii).5\}, \quad \{(i).13\};$$
$$\rho_k = 6 : \{(i).14, (i).21\}, \quad \{(i).16, (ii).9\}, \quad \{(i).19.\}, \quad \{(i).20\}, \quad \{(i).15, (i).17, (ii).11\}, \quad \{(i).18, (ii).8\}, \quad \{(ii).10\};$$
$$\rho_k = 5 : \{(i).23, (i).27\}, \quad \{(i).22\}, \quad \{(i).25\}, \quad \{(i).26\}, \quad \{(i).24, (ii).13\}, \quad \{(i).28\}, \quad \{(ii).12\}, \quad \{(ii).14)\};$$
$$\rho_k = 4 : \{(i).29\}, \quad \{(i).30\}, \quad \{(i).31\}, \quad \{(i).32\}, \quad \{(ii).15\};$$
$$\rho_k = 3 : \{(i).33\}.$$

Proof. Every equivalence above is similar to (3.2.4) or (3.2.5). In (3.2.4), we take the Cremona map of two points on the line L and one point on the conic Q. For this to be defined over $\mathbb{Q}$, these three points must be closed under the action of G, that is, we must have at least one $\mathbb{Q}$-point in the conic Q and either two $\mathbb{Q}$-points in L or a pair of conjugate points.

In (3.2.5), we take the Cremona map defined by three points of Q. Therefore, for the map to be defined over $\mathbb{Q}$, we need that all the three points to be defined over $\mathbb{Q}$, or to have one $\mathbb{Q}$-point and a pair of conjugate points, or finally that the three points form one 3-orbit. □

Remark 3.2.8. If we take the Cremona map φ over 2 points of Q and one point of L, then we will have $\deg(\varphi_*(L)) = 1$ and $\deg(\varphi_*(Q)) = 2$, so we will get a construction in the same configuration as before.

3.2.2 Explicit Examples of Cubic Pencils

In this section, our goal is to, given a configuration $(i).N$ in Table 1, realize it as an explicit cubic pencil generated by cubics defined over $\mathbb{Q}$. Assuming that this configuration has at least one base point defined over $\mathbb{Q}$, that is, $(i).N$ has at least one 1-orbit in Table 1, by Thm. 1.3.14 we can find a cubic pencil Λ over $\mathbb{Q}$ by fixing 8 points in $\mathbb{P}^2$ following the configuration of the remaining 8 base points. Then, the base point over $\mathbb{Q}$ will appear naturaly as the ninth base point of Λ. Below, we give the required steps for obtaining this construction.

Step 1: We deal first with the constructions $(i).N.$ that admit a $\mathbb{Q}$-point on a conic. For that we construct a conic through five points that satisfy the orbit configuration of $(i).N$. This task is simple as we can find a conic through any given five points $[x_1 : y_1 : z_1], ..., [x_5 : y_5 : z_5]$ such that no three are collinear. The aforementioned conic is given by the equation:

$$\det \begin{pmatrix} x^2 & y^2 & z^2 & xy & xz & yz \\ x_1^2 & y_1^2 & z_1^2 & x_1y_1 & x_1z_1 & y_1z_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & x_2z_2 & y_2z_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & x_3z_3 & y_3z_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & x_4z_4 & y_4z_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & x_5z_5 & y_5z_5 \end{pmatrix} = 0.$$

Using this, we see that it is enough to take any five points $M = \{p_1, ..., p_5\}$ such that no three are collinear and the orbits of M follow the structure of $(i).N$, and the conic Q will be the unique conic through the points of M. Since the points in M are invariant under the action of G, Q will be defined over $\mathbb{Q}$.

On the other hand, if we want to exhibit a construction such that the base point defined over $\mathbb{Q}$ lies on the line in configuration $(i).N$, then we must find an irreducible conic Q together with *six* points of Q following the Galois structure of $(i).N$. As it is not necessarily true that given six points in the plane, there is a conic through them, we must start from a conic and exhibit the points that fit the desired orbit configuration. We can do this by taking an irreducible conic Q defined over $\mathbb{Q}$ of the form:

$$Q : y^2 + axy + byz + cxz + dz^2 = 0.$$

Then, for each $\alpha \in \overline{\mathbb{Q}}$ such that $\alpha \neq -c/a$, there is a unique $x(\alpha) \in \mathbb{Q}[\alpha]$ such that $[x(\alpha) : \alpha : 1]$ is a point on Q, given by:

$$x(\alpha) = \frac{-\alpha^2 - b\alpha - d}{a\alpha + c}.$$

Taking six numbers $y_1, ..., y_6 \in \overline{\mathbb{Q}}$ with the orbit structure of $(i).N$, we find the six points $M = \{p_1, ..., p_6\} \subset Q$ given by $p_i = [x(y_i) : y_i : 1]$ following the required structure.

Step 2: Take a line L defined over $\mathbb{Q}$ such that L does not intersect M and is not tangent to Q.

If the $\mathbb{Q}$-base point lies on Q, then take three points $M' = \{p_6, p_7, p_8\} \subset L$ following the orbit structure of $(i).N$ such that $p_i\,(i = 6, 7, 8)$ is not collinear with two points in M. Notice that this is possible since there are finitely many lines passing through two points of M.

Similarly, if the $\mathbb{Q}$-base point lies on L, then we take the remaining two base points $M' = \{p_7, p_8\} \subset L$ following the orbit structure of $(i).N$, with p_7, p_8 not collinear with two points in M.

Now, by the Cayley-Bacharach Theorem (1.3.14), the eight points in $M \cup M'$ define a cubic pencil Λ such that the reducible cubic $Q \cdot L \in \Lambda$.

Step 3: To give Λ explicitly, we must find another cubic $C \in \Lambda$ defined over $\mathbb{Q}$. We do this by finding an irreducible cubic curve C through the eight points in $M \cup M'$ and one other $\mathbb{Q}$-point q *outside* of $Q \cup L$.

We can find a cubic through any nine given points $[x_1 : y_1 : z_1], ..., [x_9 : y_9 : z_9]$ such that no four are on a line and no seven are on a conic. The equation for this cubic is given by:

$$\det \begin{pmatrix} x^3 & y^3 & z^3 & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 & xyz \\ x_1^3 & y_1^3 & z_1^3 & x_1^2y_1 & x_1^2z_1 & x_1y_1^2 & y_1^2z_1 & x_1z_1^2 & y_1z_1^2 & x_1y_1z_1 \\ x_2^3 & y_2^3 & z_2^3 & x_2^2y_2 & x_2^2z_2 & x_2y_2^2 & y_2^2z_2 & x_2z_2^2 & y_2z_2^2 & x_2y_2z_2 \\ x_3^3 & y_3^3 & z_3^3 & x_3^2y_3 & x_3^2z_3 & x_3y_3^2 & y_3^2z_3 & x_3z_3^2 & y_3z_3^2 & x_3y_3z_3 \\ x_4^3 & y_4^3 & z_4^3 & x_4^2y_4 & x_4^2z_4 & x_4y_4^2 & y_4^2z_4 & x_4z_4^2 & y_4z_4^2 & x_4y_4z_4 \\ x_5^3 & y_5^3 & z_5^3 & x_5^2y_5 & x_5^2z_5 & x_5y_5^2 & y_5^2z_5 & x_5z_5^2 & y_5z_5^2 & x_5y_5z_5 \\ x_6^3 & y_6^3 & z_6^3 & x_6^2y_6 & x_6^2z_6 & x_6y_6^2 & y_6^2z_6 & x_6z_6^2 & y_6z_6^2 & x_6y_6z_6 \\ x_7^3 & y_7^3 & z_7^3 & x_7^2y_7 & x_7^2z_7 & x_7y_7^2 & y_7^2z_7 & x_7z_7^2 & y_7z_7^2 & x_7y_7z_7 \\ x_8^3 & y_8^3 & z_8^3 & x_8^2y_8 & x_8^2z_8 & x_8y_8^2 & y_8^2z_8 & x_8z_8^2 & y_8z_8^2 & x_8y_8z_8 \\ x_9^3 & y_9^3 & z_9^3 & x_9^2y_9 & x_9^2z_9 & x_9y_9^2 & y_9^2z_9 & x_9z_9^2 & y_9z_9^2 & x_9y_9z_9 \end{pmatrix} = 0.$$

Notice that since M and M' are invariant over G, the cubic C through $M \cup M' \cup \{q\}$ will be defined over $\mathbb{Q}$.

Step 4: To see that the cubic pencil $\Lambda : \lambda C + \mu(Q \cdot L)$ is an explicit example of a construction with the desired configuration, we must verify that the remaining base point of Λ is indeed defined over $\mathbb{Q}$.

The base points of this cubic pencil will be given by $B := C \cap (Q \cup L)$. We can see that $B = M \cup M' \cup \{p_0\}$, that is, the first eight chosen points and a ninth point. Since B is given by the intersection of curves defined over $\mathbb{Q}$, B must be invariant under the action of G. Consequently, noticing that M and M' were taken to be invariant under G, the point p_0 must be defined over $\mathbb{Q}$.

Example 3.2.9. Explicit example of a cubic pencil with $(i).1$ configuration:

We need to find an irreducible cubic C over $\mathbb{Q}$ such that C intersects a conic Q and a line L only in rational points.

Let Q be a conic and L a line defined by the equations:

$$\begin{aligned} Q &: \ y^2 + 2xy + 3yz + xz + z^2 = 0 \\ L &: \ x - 3y = 0. \end{aligned}$$

Now we take five $\mathbb{Q}$-points over Q and three over L:

$$\begin{aligned} M &= \{[5:3:3], [-1:-1:1], [-1:0:1], [1:6:-1], [1:-15:5]\} \\ M' &= \{[3:1:1], [6:2:3], [3:1:3]\}. \end{aligned}$$

Let q be a point such that $q \notin Q \cup L$. Here, we take $q = [1:1:1]$. Now we calculate the cubic C_1 through the points in $M \cup M' \cup \{q\}$ using the determinant:

$$\begin{aligned} C_1 : 7399x^3 - \tfrac{1662829}{45}x^2y + \tfrac{491449}{15}xy^2 + \tfrac{134197}{5}y^3 + \tfrac{1378657}{45}x^2z \\ - \tfrac{4474484}{45}xyz + \tfrac{568421}{15}y^2z + \tfrac{1122338}{45}xz^2 - \tfrac{3788512}{45}yz^2 + \tfrac{76636}{45}z^3 = 0. \end{aligned}$$

Finally, we can calculate the intersection points $C_1 \cap (Q \cup L)$ to find the last base point p_0:

$$\begin{aligned} C_1 \cap Q &= \left\{ \begin{matrix} [5:3:3], & [-1:-1:1], & [-1:0:1], \\ [1:6:-1], & [1:-15:5], & [\frac{-1495489}{327217} : \frac{-1159}{2167} : 1] \end{matrix} \right\} \\ C_1 \cap L &= \{[3:1:1], [6:2:3], [3:1:3]\}. \end{aligned}$$

The cubics C_1 and $Q \cdot L$ define a cubic pencil $\lambda C_1 + \mu(Q \cdot L)$ with configuration $(i).1$. Then, this pencil gives rise to an elliptic surface S with $\rho_{\mathbb{Q}} = 10$ and $r_{\mathbb{Q}} = 7$.

Example 3.2.10 ((i).2)**.** Through the same process, we create a cubic pencil with a cubic C_2 and the same conic and line Q and L.

$$\begin{aligned} Q &: \ y^2 + 2xy + 3yz + xz + z^2 = 0 \\ L &: \ x - 3y = 0. \end{aligned}$$

$$C_2 : \tfrac{1099}{3}x^3 - \tfrac{41068}{25}x^2y + \tfrac{18097}{15}xy^2 + \tfrac{26056}{25}y^3 + \tfrac{84943}{75}x^2z$$
$$-\tfrac{284956}{75}xyz + \tfrac{41719}{25}y^2z + \tfrac{20444}{25}xz^2 - \tfrac{68416}{25}yz^2 + \tfrac{1288}{25}z^3 = 0.$$

The base points of the pencil are given by:

$$C_2 \cap Q = \begin{Bmatrix} [-1 : -1 : 1], & [-1 : 0 : 1], & [1 : -15 : 5] \\ [\tfrac{-3i}{5} - \tfrac{6}{5} : i : 1], & [\tfrac{3i}{5} - \tfrac{6}{5} : -i : 1], & [\tfrac{-789119}{235029} : \tfrac{-827}{1497} : 1] \end{Bmatrix}$$
$$C_2 \cap L = \{[3 : 1 : 1], [6 : 2 : 3], [3 : 1 : 3]\}.$$

This pencil will give rise to a $\mathbb{Q}$-rational surface with $\rho_{\mathbb{Q}} = 9$ and $r_{\mathbb{Q}} = 6$.

Example 3.2.11 ((i).4)**.** We create a cubic pencil generated by a cubic C_3 and $Q \cdot L$.

$$C_3 : -684x^3 + \tfrac{260334}{85}x^2y - \tfrac{182463}{85}xy^2 - \tfrac{162981}{85}y^3 - \tfrac{124449}{85}x^2z +$$
$$\tfrac{461052}{85}xyz - \tfrac{346923}{85}y^2z - \tfrac{66309}{85}xz^2 + \tfrac{219879}{85}yz^2 = 0.$$

The base points of the pencil are given by:

$$C_3 \cap Q = \begin{Bmatrix} [\tfrac{-18i}{17} - \tfrac{21}{17} : 2i : 1], & [\tfrac{18i}{17} - \tfrac{21}{17} : -2i : 1], & [-1 : 0 : 1] \\ [\tfrac{-3i}{5} - \tfrac{6}{5} : i : 1], & [\tfrac{3i}{5} - \tfrac{6}{5} : -i : 1], & [\tfrac{-62245}{22857} : \tfrac{-229}{401} : 1] \end{Bmatrix}$$
$$C_3 \cap L = \{[0 : 0 : 1], [3 : 1 : 1], [3 : 1 : 3]\}.$$

This pencil will give rise to a $\mathbb{Q}$-rational surface with $\rho_{\mathbb{Q}} = 8$ and $r_{\mathbb{Q}} = 5$.

Example 3.2.12 ((i).10)**.** We create a cubic pencil generated by a cubic C_4 and $Q \cdot L$.

$$C_4 : 11664x^3 - \tfrac{249318}{5}x^2y + \tfrac{2752947}{85}xy^2 + \tfrac{2552229}{85}y^3 + \tfrac{1800873}{85}x^2z$$
$$-\tfrac{7266672}{85}xyz + \tfrac{5592159}{85}y^2z + \tfrac{809433}{85}xz^2 - \tfrac{2994003}{85}yz^2 = 0.$$

The base points of the pencil are:

$$C_4 \cap Q = \begin{Bmatrix} [\tfrac{-18i}{17} - \tfrac{21}{17} : 2i : 1], & [\tfrac{18i}{17} - \tfrac{21}{17} : -2i : 1], & [-1 : 0 : 1] \\ [\tfrac{-3i}{5} - \tfrac{6}{5} : i : 1], & [\tfrac{3i}{5} - \tfrac{6}{5} : -i : 1], & [\tfrac{-439}{192} : \tfrac{-19}{32} : 1] \end{Bmatrix}$$
$$C_4 \cap L = \{[0 : 0 : 1], [3i : i : 1], [-3i : -i : 3]\}.$$

This pencil will give rise to a $\mathbb{Q}$-rational surface with $\rho_{\mathbb{Q}} = 7$ and $r_{\mathbb{Q}} = 4$.

Example 3.2.13 ((i).16)**.** Take the cubic C_5, the conic Q_1 and the line L_1 given by:

$$C_5 : x^3 + 5xy^2 - x^2z - 2y^2z + 3xz^2 = 0;$$
$$Q_1 : 2x^2 + y^2 - 3z^2 = 0;$$
$$L_1 : x = z.$$

The base points of the pencil $\lambda C_5 + \mu Q_1 \cdot L_1$ are:

$$C_5 \cap Q_1 = \left\{ \begin{matrix} [\sqrt{2} : i : 1], & [\sqrt{2} : -i : 1], & [1 : 5 : 3] \\ [-\sqrt{2} : i : 1], & [-\sqrt{2} : -i : 1], & [1 : -5 : 1] \end{matrix} \right\}$$
$$C_5 \cap L_1 = \{[1 : i : 1], [1 : -i : 1], [0 : 1 : 0]\}.$$

This pencil gives rise to a $\mathbb{Q}$-rational surface with $\rho_{\mathbb{Q}} = 6$ and $r_{\mathbb{Q}} = 4$.

3.3 Rational Elliptic Surfaces that are $\mathbb{Q}$-irrational

In this section, we will give an example of a rational elliptic surface that is not $\mathbb{Q}$-rational. We do this by blowing up a del Pezzo surface of degree 2. First, we give a characterization of these surfaces.

Definition 3.3.1. We say that a surface X is a *double cover* of $\mathbb{P}^2$ *ramified* over a smooth plane curve C if there exists a morphism $\varphi : X \to \mathbb{P}^2$ such that:

1. For every $P \in \mathbb{P}^2 \backslash C$, $\varphi^{-1}(P)$ is given by two points in X;
2. For $P \in C$, $\varphi^{-1}(P)$ is given by a single point in X.

Theorem 3.3.2. *Let X be a surface defined over $\mathbb{Q}$. Then, X is a del Pezzo surface of degree 2 if and only if X is a double cover of $\mathbb{P}^2$ ramified over a smooth plane quartic defined over $\mathbb{Q}$.*

Proof. See [Kuw05, Prop. 3.1, 3.2], Propositions 3.1 and 3.2. □

Let $C \subset \mathbb{P}^2$ be a smooth plane quartic curve, given by:

$$C : \; F(x, y, z) = 0.$$

Then, the degree 2 del Pezzo surface X can be characterized as a double cover of the plane ramified over C by the affine equation:

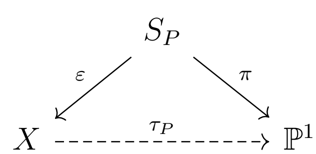
$$X : \; w^2 = F(x, y, 1).$$

Take a point P in $\mathbb{P}^2$, and let $L(x,y,z), R(x,y,z)$ be two different lines passing though P. Then, L and R generate a pencil of lines $\lambda L + \mu R$. The pullback of any line in this pencil by φ, the curve $\Gamma_{u,v} := \phi^*(uL + vR)$, is a double cover of $\mathbb{P}^1$ ramified over 4 points. Then, by Hurwitz Theorem (see [Har77, IV.2.4]), we know that $\Gamma_{u,v}$ is a genus 1 curve.

Let ψ_P be a rational map defined by:

$$\begin{aligned} \psi_P : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ Q &\mapsto [L(Q) : R(Q)]. \end{aligned}$$

The map ψ_P is defined everywhere, except on P. Now, let $\tau_P : X \dashrightarrow \mathbb{P}^1$ be the rational map given by the composition $\tau_P = \psi_P \circ \varphi$. The map τ_P is not defined only over $\varphi^{-1}(P)$. Then, resolving the indeterminate points by blowing-up, we get a surface S_P with a morphism $\pi : S_P \to \mathbb{P}^1$.



The fiber of π over a point $[u : v]$ is isomorphic to $\Gamma_{-v,u}$ for almost every $[u : v] \in \mathbb{P}^1$. Then, if P is defined over $\mathbb{Q}$, S_P is an elliptic surface over $\mathbb{Q}$.

The geometry of S_P will depend on the choice of P. If $P \notin C$, then $\varphi^{-1}(P) = \{\tilde{P}_1, \tilde{P}_2\}$, and S_P is the blow-up of X at $\tilde{P}_1$ and $\tilde{P}_2$.

If $P \in C$, then $\varphi^{-1}(P) = \{\tilde{P}\}$, and S_P will come from two blow-ups over $\tilde{P}$, that is, one blow-up over $\tilde{P}$ and then another at a point of the exceptional divisor E. Furthermore, the singular fibers of $\pi : S_P \to \mathbb{P}^1$ depend on the tangent T_P of the quartic C at P.

Theorem 3.3.3. *Let $P \notin C$ be a point of $\mathbb{P}^2$ such that all of the lines through P in $\lambda L + \mu R$ intersect the quartic C in at least 3 different points. Then, the elliptic surface S_P coming from the double cover $\varphi : X$ ramified at C will be a rational elliptic surface with Mordell-Weil rank 8 over $\overline{\mathbb{Q}}$.*

Proof. See [Kuw05, 4.1]. □

Theorem 3.3.4. *Let $P \in C$ be a point in a smooth quartic and T_P be the tangent line of C through P. If $T_P \cap C = \{P, P', P''\}$, with $P' \neq P''$, then the elliptic surface S_P coming from the double cover $\varphi : X \to \mathbb{P}^2$ ramified at C will be a rational elliptic surface with Mordell-Weil rank 7 over $\overline{\mathbb{Q}}$.*

Proof. See [Kuw05, 4.2]. □

Notice that when the degree 2 del Pezzo X has $Pic(X)_{\mathbb{Q}} = \mathbb{Z}$, X is a $\mathbb{Q}$-minimal model of S_P and, by (1.4.25), S_P is not $\mathbb{Q}$-rational.

Example 3.3.5. If X is a degree 2 del Pezzo with $Pic(X) = \mathbb{Z}$, then the blow-up of X in the points $P_1, P_2 \in \varphi^{-1}(P)$ gives us a rational elliptic surface S_P. This surface will be an example of a rational elliptic surface with Mordell-Weil rank 8 over $\overline{\mathbb{Q}}$ that is not $\mathbb{Q}$-rational.

Example 3.3.6. Let X be the degree 2 del Pezzo given by the affine equation:

$$w^2 = x^3 + A(y)x^2 + B(y)x + C(y)$$

where A, B, C are polynomials of degree $2, 3$ and 4, respectively, given by:

$$\begin{aligned} A(y) &= y^2 + 1; \\ B(y) &= y^3 + 7y^2 - 5y; \\ C(y) &= y^4 - 9y^3 + y + 1. \end{aligned}$$

Then, blowing up the point $P = [1 : 0 : 0 : 0]$, we get an elliptic surface $\pi : S_P \to \mathbb{P}^1$.

3.4 Concluding remarks

In this chapter, we have seen that rational elliptic surfaces over the rational numbers with geometric Mordell-Weil rank 7 can be constructed in many different ways, specially in contrast to the complex case. Even when the surfaces arise from the blow-up at the base points of a pencil of cubics defined over $\mathbb{Q}$, simulating Thm. 2.3.1 for algebraically closed fields, there are a lot of possible structures of Galois orbits in the base points (as seen in Tables 1 and 2), and few of them are equivalent (see 3.2.7). We have seen different explicit examples of $\mathbb{Q}$-rational elliptic surfaces, including one that shows us that the condition in Thm. 2.3.8 for when a rational surface is $\mathbb{Q}$-rational is not necessary (see Ex. 3.2.13). We have also seen an example of a rational elliptic surface that is not $\mathbb{Q}$-rational (see Ex. 3.3.6).

Moving forward, some possible future points of study in this area are:

1. Finding examples of cubic pencils in every possible configuration from tables (i) and (ii).

2. Determining the field of definition of the full Mordell-Weil group of a rational elliptic surface with given geometric Mordell-Weil rank.

3. Finding examples of $\mathbb{Q}$-rational elliptic surfaces that do not arise from the base points of a cubic pencil.

4. Defining a notion of equivalence of constructions for any construction of a rational elliptic surface over $\mathbb{Q}$.

5. Finding $\mathbb{Q}$-equivalent constructions for rational elliptic surfaces with other geometric Mordell-Weil ranks.

6. Finding the possible constructions of rational elliptic surfaces over $\mathbb{C}$ with Mordell-Weil rank at most 3.

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